SUPER LOG-CONCAVITY OF THE FIRST EIGENFUNCTIONS FOR HORO-CONVEX DOMAINS IN HYPERBOLIC SPACE

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ABSTRACT. In this paper, we prove that the first eigenfunction of the Laplacian for a horo-convex domain $\Omega \subset \mathbb{H}^n$ is super log-concave when $\operatorname{diam}(\Omega)$ is not large. Our result is optimal in the sense that there are counterexamples when Ω is not horo-convex or when $\operatorname{diam}(\Omega)$ is large respectively.

1. INTRODUCTION

Given a bounded smooth connected domain $\Omega \subset M^n$ of a Riemannian manifold, the eigenvalue equation of the Laplacian in Ω with the Dirichlet boundary condition is

$$\triangle\psi = -\mu\psi, \ \psi|_{\partial\Omega} = 0.$$

The eigenvalues $\{\mu_i | i = 1, 2, \cdots\}$ satisfy $0 < \mu_1 < \mu_2 \le \mu_3 \le \cdots \to \infty$. Consider the first eigenvalue equation

$$\begin{aligned} \triangle v + \mu_1 v = 0 \qquad & \text{in } \Omega \\ v = 0 \quad & \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

In this paper, we will study the log-concavity property of the first eigenfunction $v > 0$ in hyperbolic space.

The study of log-concavity of the first eigenfunction for the Laplacian has a long history. In 1976, Brascamp-Lieb [3] proved that if v satisfies (1.1) and Ω is a convex domain in $\mathbb{R}^n$, then $\log v$ is concave. Later, Caffarelli-Spruck [5] gave an elementary proof of this result using the idea of [14]. When Ω is a convex domain in $\mathbb{S}^n$, applying the continuity method developed in [20], Lee-Wang proved the log-concavity of the first eigenfunction in [16]. In [11], the first author and her collaborators obtained the log-concavity of the first eigenfunction for convex domains of surfaces with positive sectional curvature.

Notice that the results mentioned above all require that the ambient manifold M^n has nonnegative sectional curvature. One may ask whether the first eigenfunction of a convex domain is log-concave if not all sectional curvatures of the ambient manifold M^n are nonnegative. Unfortunately, the answer to this question is negative even when M^n is the hyperbolic space [19]. In fact, it is shown in [2] that there exists a convex domain $\Omega \subset \mathbb{H}^n$ such that the first eigenfunction v of Ω is not log-concave in any sense. That is, the largest Hessian eigenvalue of $\log v$ goes to positive infinity. This is also the case even for manifolds that have only a single tangent plane of negative sectional curvature [10]. Hence, to prove the log-concavity of the first eigenfunction of Ω in a manifold M^n that contains tangent planes of negative sectional curvature, we need more restrictions on Ω.

To start, let us study the model case, that is, when M^n is the hyperbolic space $\mathbb{H}^n$. For $\mathbb{H}^n$, a natural stronger convexity to consider is horo-convexity. Recall that a domain $\Omega \subset \mathbb{H}^n$ is said to be **horo-convex**, if for every point $p \in \partial\Omega$, Ω lies on the convex side of some horosphere $S_h(p)$ through p. When Ω has a smooth boundary, it is equivalent to the second fundamental form of $\partial\Omega$ satisfying $\mathrm{II} \ge I_{n-1}$, that is, the principal curvatures of $\partial\Omega$ are ≥ 1.

Indeed, we prove that the first eigenfunction of horo-convex domains is log-conacve when the diameter of the domain is not too big. In fact, we prove a stronger result. To state it, we need the following definition.

Definition 1. *We say that a function f is **$\boldsymbol{\lambda}$-log-concave** if the matrix $(-(\log f)_{ij} - \lambda|\nabla \log f|\delta_{ij})$ is positive semi-definite; **strictly $\boldsymbol{\lambda}$-log-concave** if the matrix $(-(\log f)_{ij} - \lambda|\nabla \log f|\delta_{ij})$ is positive definite. We also call a 1-log-concave function a **super log-concave** function.*

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Here is our main result.

Theorem 1.1. *Let $\Omega \subset \mathbb{H}^n$ be a horo-convex domain with diameter $diam(\Omega) \leq c_0$, where $c_0 = c_0(n) > 0$ is a positive constant that only depends on n. Let v be the solution of the eigenvalue problem* (1.1), *then v is super log-concave. In particular v is log-concave.*

Our result is the first log-concavity estimate for the first eigenfunction of domains in a manifold with negative sectional curvatures without requiring the radial symmetry of the domain. The result is optimal in the following sense. First, as we mentioned before, there are examples that show that the first eigenfunction of convex domains in $\mathbb{H}^n$ does not have to be log-concave. Second, when $\Omega \subset \mathbb{H}^3$ is a geodesic ball, one can numerically show that when $\mathrm{diam}(\Omega) > c_0$ for some $c_0 \in (2, 3)$, the first eigenfunction is not super log-concave. More details will be given in Section 2.

With log-concavity of the first eigenfunction, by [20], one can reduce the estimate of the Dirichlet fundamental gap to the Neumann one. Hence combining the log-concavity of v obtained in Theorem 1.1 with [6, Theorem 3] we get (see the proof of [11, Corollary 2.2] for more details)

Corollary 1.2. *Let $\Omega \subset \mathbb{H}^n$ be a horo-convex domain with diameter $D := diam(\Omega) \leq c_0$ for some $c_0 = c_0(n) > 0$, $\mu_i (i = 1, 2)$ be the first two eigenvalues of the Laplacian on Ω with Dirichlet boundary condition. Then*

$$\mu_2 - \mu_1 \geq \tfrac{\pi^2}{D^2} e^{-c(n)D}.$$

This result improves the estimate in [12] but is weaker than the estimate in [13] when the diameter D goes to zero. One can also use [1, Theorem 14] to get a lower bound for $\mu_2 - \mu_1$ in terms of a one-dimensional model. Note that here we only used the log-concavity of the first eigenfunction. We expect a better estimate with proper utilization of the super log-concavity.

To prove our result, the key is to prove the super log-concavity of the first eigenfunction in the sense of Definition 1. This allows us to utilize the horo-convexity of the domain. In particular, the extra barrier function $|\nabla \log v|$ nicely contributes to controlling the "bad" terms caused by the negative curvature of the manifold. With this setup, we use the idea of the proof of constant rank theorems, which has been used previously by various authors (see [4, 15, 7] for examples). More precisely, we apply the continuity method as a deformation process together with the strong maximum principle to force the convexity.

The organization of this paper is as follows. In Section 2, we show that the first eigenfunction of a geodesic ball is super-log-concave when the radius of the ball is not large. In Section 3, we revisit some results in [9]. We will use these results in Section 4. Following the idea of [7], we prove Theorem 1.1 in Section 4.

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2. Super log-concavity of the first eigenfunction for balls

In this section we will show that the first eigenfunction of the geodesic ball in $\mathbb{H}^n$ is super log-concave when the radius of the ball is not large. Let $\mathbb{S}^{n-1}$ be the unit sphere in Euclidean space $\mathbb{R}^n$ with the standard induced metric dz^2, then the metric of $\mathbb{H}^n$ is

$$g = dr^2 + \sinh^2 r dz^2.$$

Let $B_r \subset \mathbb{H}^n$ be a geodesic ball in $\mathbb{H}^n$ with radius r. We denote the first eigenfunction of B_r by v_B. It is easy to see that B_r is a strictly horo-convex domain for all $r > 0$. We will show the following.

Lemma 2.1. *There exists a constant $r_0 = r_0(n) > 0$ that depends only on n such that for all $r < r_0$, the first eigenfunction v_B of $B_r \subset \mathbb{H}^n$ is strictly 1-log concave.*

Proof. We will follow the proof of Lemma 5.2 in [17]. Note that by separation of variables, v_B is a radial function. Denote $\varphi = (\log v_B)'$ then φ satisfies

$$\varphi' = -\frac{n-1}{\tanh t}\varphi - \mu_1 - \varphi^2, \tag{2.1}$$

and

$$\varphi'' = \frac{n-1}{\sinh^2 t}\varphi - \frac{n-1}{\tanh t}\varphi' - 2\varphi\varphi'. \tag{2.2}$$

Here, μ_1 is the first eigenvalue of B_r. We already know $\varphi(0) = 0$ and $\varphi < 0$ on $(0, r)$. In the following, we want to show when $r < r_0$, the matrix

$$(\nabla_{ij}(\log v_B) + |\nabla \log v_B|\delta_{ij})$$

is negative definite for all $t \in (0, r)$. This is equivalent to showing that φ satisfies

$$(\coth t)\varphi - \varphi < 0 \tag{2.3}$$

and

$$\varphi' - \varphi < 0. \tag{2.4}$$

It is clear that for all $r > 0$, the inequality (2.3) always holds. We only need to look at inequality (2.4). In view of (2.1) we get

$$\varphi'(0) = -\mu_1 - (n-1)\lim_{t\to 0}\frac{\varphi}{\tanh t} = -\mu_1 - (n-1)\varphi'(0).$$

This gives $\varphi'(0) = -\frac{\mu_1}{n}$.

Now denote $g := \varphi' - \varphi$, then we have $g(0) = -\frac{\mu_1}{n} < 0$. We want to show there exists a $r_0 = r_0(n) > 0$ such that when $r < r_0$, $g(t) < 0$ on $[0, r)$. If not, then for any $r > 0$ there would exist some $t_1 \in (0, r)$ such that $g(t_1) = 0$, $g(t) < 0$ on $[0, t_1)$, and $g'(t_1) \geq 0$. Note that when $g = 0$ we have $\varphi' = \varphi$. Combining with (2.1) we get, at t_1,

$$-(n-1)(\coth t_1)\varphi - \mu_1 - \varphi^2 = \varphi.$$

Denote $A := 1 + (n-1)\coth t_1$, then

$$\varphi(t_1) = \frac{-A \pm \sqrt{A^2 - 4\mu_1}}{2}. \tag{2.5}$$

By (2.1) and (2.2) we also obtain

$$\begin{aligned} g' &= \varphi'' - \varphi' \\ &= \frac{n-1}{\sinh^2 t}\varphi - \frac{n-1}{\tanh t}\varphi' - 2\varphi\varphi' - \left(-\frac{n-1}{\tanh t}\varphi - \mu_1 - \varphi^2\right). \end{aligned}$$

In particular, at t_1 we have

$$g'(t_1) = \frac{n-1}{\sinh^2 t_1}\varphi + \mu_1 - \varphi^2.$$

Therefore, $g'(t_1) \geq 0$ is equivalent to

$$\varphi^2 - \frac{n-1}{\sinh^2 t_1}\varphi - \mu_1 \leq 0. \tag{2.6}$$

Denoting $B := \frac{n-1}{2\sinh^2 t_1}$, then inequality (2.6) holds iff

$$B - \sqrt{\mu_1 + B^2} \leq \varphi(t_1) \leq B + \sqrt{\mu_1 + B^2}.$$

Since φ is negative, this is equivalent to

$$|\varphi(t_1)| \leq \sqrt{\mu_1 + B^2} - B = \frac{\mu_1}{B + \sqrt{B^2 + \mu_1}}.$$

Take $\varphi(t_1) = \frac{-A + \sqrt{A^2 - 4\mu_1}}{2}$ in (2.5), the above is equivalent to

$$\frac{2}{A + \sqrt{A^2 - 4\mu_1}} \leq \frac{1}{B + \sqrt{\mu_1 + B^2}}.$$

By simple algebraic computations we can see, there exists a constant $r_0 = r_0(n) > 0$ which satisfies the following property: For every $r < r_0$ and $t_1 \in (0, r)$ we have

$$\frac{2}{A + \sqrt{A^2 - 4\mu_1}} > \frac{1}{B + \sqrt{\mu_1 + B^2}}.$$

This finishes the proof of the Lemma. □

2.1. **Numerical computation in $B_r \subset \mathbb{H}^3$.** When $n = 3$, all the eigenvalues and eigenfunctions of the balls can be explicitly computed; see, e. g. [17, (3)]. The first eigenfunction of B_r is

$$v_B = (\sinh t)^{-1} \sin \frac{\pi}{r} t.$$

Let $\Phi = \log v_B$, below we give a numerically computed graph of $\Phi'' - \Phi'$ for $r = 1, 2, \cdots, 9$.[1] This graph shows that when $r > c_0$ for some $c_0 \in (2, 3)$, the first eigenfunction v_B of B_r is no longer super log-concave.

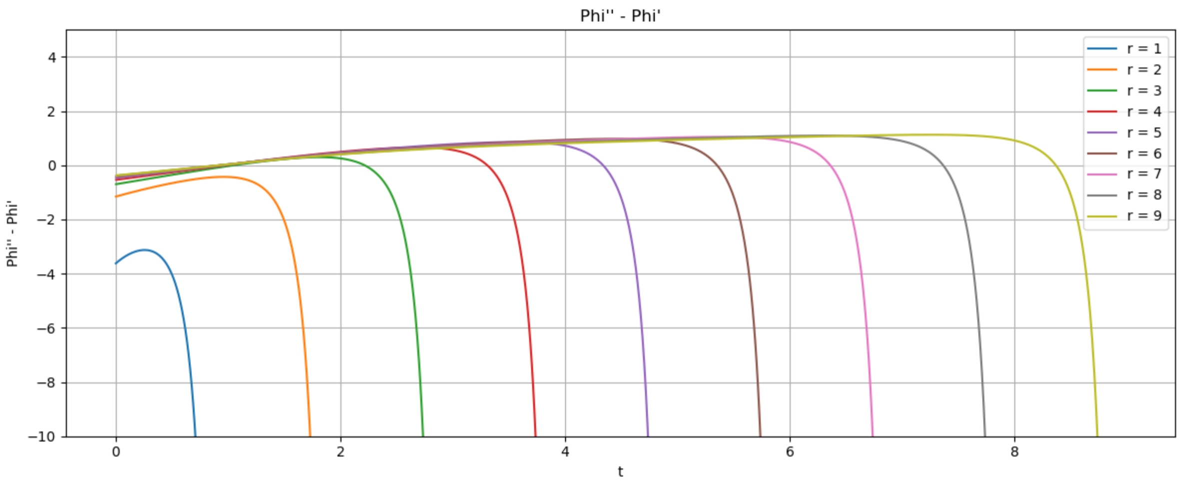

FIGURE 1. Graph of $\Phi'' - \Phi'$

3. DEFORMATION

In this section, we will provide a method that deforms a horo-convex hypersurface in the hyperbolic space into a geodesic sphere. Note that, there are many methods to do so, here we provide one that is convenient for later use.

In [9], the authors studied the following flow

$$X_t = \left[(\cosh r - u) \frac{E_{m-1}(\tilde{\kappa})}{E_m(\tilde{\kappa})} - u \right] \nu, \ m = 1, \cdots, n, \tag{3.1}$$

where ν is the unit outward normal of $M_t = X(M, t)$, $u = \langle \sinh r \, \partial_r, \nu \rangle$ is the support function, $\tilde{\kappa}_i = \kappa_i - 1$ are the shifted principal curvatures of M_t, and

$$E_m(\tilde{\kappa}) = \frac{1}{\binom{n}{m}} \sum_{1 \le i_1 \le \cdots \le i_k \le n} \tilde{\kappa}_{i_1} \cdots \tilde{\kappa}_{i_k}$$

is the normalized m-th elementary symmetric function of $\tilde{\kappa}$. For each $m = 1, \cdots, n$, let

$$\Gamma_m^+ = \{ x \in \mathbb{R}^n : E_i(x) > 0, i = 1, \cdots, m \}$$

be the Garding cone. They proved

Theorem 3.1 (Theorem 1.6 of [9])**.** *Let $X_0 : M_n \to \mathbb{H}^{n+1} (n \ge 2)$ be a smooth embedding such that $M_0 = X_0(M)$ is a smooth, horo-convex hypersurface in $\mathbb{H}^{n+1}$ with $\tilde{\kappa} \in \Gamma_m^+$. The flow (3.1) then has a smooth solution for all time $t \in [0, \infty)$, and $M_t = X_t(M)$ is strictly horo-convex for each $t > 0$ and converges smoothly and exponentially to a geodesic sphere.*

They also proved

[1]Figure 1 is graphed by Matthew McGonagle using Matplotlib [8].

Proposition 3.2 (Proposition 6.1 of [9]). *Let $\varphi \in C^\infty(\mathbb{S}^n \times [0,\infty))$ be a smooth solution to*

$$\varphi_t = \left((\cosh r - u)\frac{E_{m-1}(\tilde\kappa)}{E_m(\tilde\kappa)} - u \right) \frac{\sqrt{1+|D\varphi|^2}}{\sinh r}.$$

Then

$$\min_{\theta\in\mathbb{S}^n} \varphi(\cdot,0) \le \varphi(\cdot,t) \le \max_{\theta\in\mathbb{S}^n} \varphi(\cdot,0).$$

This proposition implies that if $M_t = \{(\rho(\theta,t),\theta) : (\theta,t) \in \mathbb{S}^n \times [0,\infty)\}$ is the family of flow hypersurfaces, then

$$\min_{\theta\in\mathbb{S}^n} \rho(\cdot,0) \le \rho(\cdot,t) \le \max_{\theta\in\mathbb{S}^n} \rho(\cdot,0).$$

We will use these results in Section 4 to deform the horoconvex domain Ω to a geodesic ball.

4. Log-concavity of the first eigenfunction

In this section, we will prove our main result Theorem 1.1. The idea of the proof follows [7].

4.1. **Set up.** Let $B_{\rho_0} \subset \mathbb{H}^n$ be a geodesic ball of radius ρ_0 in $\mathbb{H}^n$. We will choose $r_0 > \rho_0 > 0$ such that the first eigenvalue of B_{ρ_0}, which is denoted by $\mu_1(B)$, satisfying $\mu_1(B) \ge c^* := c^*(n) > n^2/4$. Here, r_0 is the constant determined in Lemma 2.1 and $c^* > n^2/4$ is a fixed constant that only depends on the dimension n. It's clear that, there exists a $c_0 = c_0(n) > 0$ such that the above condition can be satisfied as long as $\rho_0 \le c_0$.

Now, letting $\Omega \subset B_{\rho_0}$ be a horo-convex domain containing the origin, we can flow $M_0 = \partial\Omega$ by (3.1). Then applying Theorem 3.1, we know that M_t is strictly horo-convex for $t > 0$ and converges smoothly and exponentially to a geometric sphere of radius r_∞, that is ∂B_{r_∞}. Moreover, by Proposition 3.2 we also have $M_t \subset B_{\rho_0}$, thus $r_\infty \le \rho_0$. This guarantees the first eigenfunction of B_{r_∞} is strictly 1-log-concave and $\mu_1(B_{r_\infty}) \ge c^*$. Moreover, using Ω_t to denote the domain inclosed by M_t, then for all $t \in [0,\infty)$ we have $\mu_1(\Omega_t) \ge c^*$.

We want to point out that when $\Omega \subset B^2_{\rho_0} \subset \mathbb{H}^2$ is a 2 dimensional domain, we can always find a horo-convex domain $\hat\Omega \subset B^3_{\rho_0} \subset \mathbb{H}^3$ such that $\hat\Omega \cap \{\theta_2 = 0\} = \Omega$. Then we can apply Theorem 3.1 to $\partial\hat\Omega$ which leads to a deformation of $\partial\Omega$.

We also want to point out that when there exists some point $p \in \partial\Omega$ such that $\tilde\kappa(p) \notin \Gamma_1^+$, that is, when M_0 cannot be flowed by (3.1), we can approximate Ω uniformly in C^2 by smooth horo-convex domains Ω^ϵ satisfying $\tilde\kappa \in \Gamma_1^+$ for all $p \in \partial\Omega^\epsilon$. We apply Theorem 3.1 to $\partial\Omega^\epsilon$ and obtain that the first eigenfunction of Ω^ϵ is super log-concave. We then let ϵ tend to 0 to complete the proof of Theorem 1.1. Therefore, in the following, without loss of generality, we will always assume that $M_0 = \partial\Omega$ is a valid initial hypersurface for the flow (3.1).

4.2. **Super log-concavity.** In this subsection we will show that for all $t > 0$, the first eigenfunction of Ω_t is strictly 1-log-concave, that is, the matrix

$$(-(\log v_{\Omega_t})_{ij} - |\nabla \log v_{\Omega_t}|\delta_{ij})$$

is positive definite, where v_{Ω_t} is the first eigenfunction of Ω_t. Since $\partial\Omega_t \to \partial\Omega$ as $t \to 0$, by continuity, we will obtain the first eigenfunction of Ω is 1-log-concave, that is, super log-concave.

Recall that $\partial\Omega_t \to \partial B_{r_\infty}$ smoothly and exponentially, by continuity we know when $t > T$ for some $T > 0$ sufficiently large, the first eigenfunction of Ω_t is strictly 1-log-concave. Now we decrease t and assume $t^* \in (0,T]$ to be the first time such that $(-(\log v_{\Omega_{t^*}})_{ij} - |\nabla \log v_{\Omega_{t^*}}|\delta_{ij})$ is not positive definite in Ω_{t^*}. Moreover, without loss of generality, we will always assume $x_0 \in \bar\Omega_{t^*}$ to be the "most degenerate" point in $\bar\Omega_{t^*}$, i.e., the eigenvalues of $(-(\log v_{\Omega_{t^*}})_{ij} - |\nabla \log v_{\Omega_{t^*}}|\delta_{ij})$ contain the most zeros at x_0. In the following, we will show that x_0 does not exist in two steps:

1. We will show that if x_0 exists then it's away from $\partial\Omega_{t^*}$.
2. We will show that x_0 cannot be an interior point of Ω_{t^*}.

For simplicity, we will drop the subscript and write v, Ω instead of $v_{\Omega_{t^*}}$, Ω_{t^*}. Let us denote $u := -\log v$, then u satisfies

$$\begin{cases} \triangle u = \mu_1 + w^2 & \text{in } \Omega \\ u \to +\infty & \text{as } x \to \partial\Omega, \end{cases} \tag{4.1}$$

where

$$w = |\nabla u|.$$

We will show that the matrix $(u_{ij} - w\delta_{ij})$ is positive definite.

4.2.1. *Strict 1-log-concavity near $\partial\Omega_t$.* For convenience, we use the following definition.

Definition 2. *We say that a smooth domain $\Omega \subset \mathbb{H}^n$ is **$\boldsymbol{\lambda}$-convex** if the second fundamental form of its boundary $\partial\Omega$ satisfies $\mathrm{II} \geq \lambda I_{n-1}$; **strictly $\boldsymbol{\lambda}$-convex** if $\mathrm{II} > \lambda I_{n-1}$.*

By Theorem 3.1 we know that for any $t > 0$, there exists a $\lambda > 1$ such that the domain Ω_t is λ-convex. In the following, we will show that the first eigenfunction of the above domain is strictly 1-log-concave near the boundary.

Lemma 4.1. *Given a strictly λ-convex domain $\Omega \subset \mathbb{H}^n$ for $\lambda \in \mathbb{R}_+$, let v be the solution of* (1.1)*, then v is λ-log-concave in a small neighborhood of $\partial\Omega$.*

We can follow the proof in [18, Lemma 3.4].

Proof. Let ν denote the unit outward normal of $\partial\Omega$. Then for any $p \in \partial\Omega$, $\nabla v|_p = -\|\nabla v\|\nu_p$. Now for any $e \in T_p\partial\Omega$ such that $e \perp \nu$, we have

$$\mathrm{Hess}\, v(e,e) = \mathrm{II}(e,e)\nabla_\nu v = -\mathrm{II}(e,e)|\nabla v|, \tag{4.2}$$

and

$$\mathrm{Hess}\log v(e,e) + \lambda|\nabla \log v|\|e\|^2 = \frac{1}{v}\left(\mathrm{Hess}\, v(e,e) - \tfrac{|\nabla_e v|^2}{v} + \lambda|\nabla v|\|e\|^2\right). \tag{4.3}$$

Write $e = e^T + e^\perp$, where $e^\perp = \langle e, \frac{\nabla v}{\|\nabla v\|}\rangle \frac{\nabla v}{\|\nabla v\|}$. Then $|\nabla_e v|^2 = \langle e, \nabla v\rangle^2 = |\nabla v|^2|e^\perp|^2$. Since as $x \to \partial\Omega$, $v \to 0$ and $|\nabla v| \neq 0$, we have $\mathrm{Hess}\log v(e,e) + \lambda|\nabla \log v|\|e\|^2 \to -\infty$ when $e^\perp \neq 0$.

For $e = e^T$, from (4.2) and(4.3),

$$\mathrm{Hess}\log v(e,e) + \lambda|\nabla \log v|\|e\|^2 = \frac{1}{v}\left(-\mathrm{II}(e,e)|\nabla v| + \lambda|\nabla v|\|e\|^2\right). \tag{4.4}$$

Since Ω is strictly λ-convex, we can see that (4.4) is negative. By continuity, we know that v is λ-log-concave in a small neighborhood of $\partial\Omega$. $\square$

4.2.2. *Strictly 1-log-concavity in Ω_t.* By the discussion from Subsection 4.2.1 we know that the degenerate point x_0 of the matrix $(u_{ij} - w\delta_{ij})$, if it exists, must be away from $\partial\Omega$. In this subsection, we show that x_0 cannot be an interior point of Ω. In other words, x_0 does not exist.

Lemma 4.2. *Let $U \subset \Omega$ be a neighborhood of the degenerate point x_0, and $u \in C^\infty(\Omega)$ a solution of* (4.1)*. Suppose the matrix $\Lambda = (\Lambda_{ij})$, where $\Lambda_{ij} := u_{ij} - w\delta_{ij}$, is positive semi-definite in U. Then Λ is positive definite in U.*

Proof. For $x_0 \in U$, we may choose a local orthonormal frame $\{e_1, \cdots, e_n\}$ at x_0 so that Λ is diagonal at x_0, and $\Lambda_{ii} = \lambda_i$ for $i = 1, \cdots, n$. We may also assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Note that, by our assumption we have $\lambda_n(x_0) = 0$.

First, we assume $w(x_0) = |\nabla u(x_0)| = 0$. Let $\mathbf{v}$ be a vector field obtained by parallel translating e_1 in a neighborhood U of x_0. Consider the matrix $\tilde{\Lambda}_{ij} := u_{ij} - \langle \nabla u, \mathbf{v}\rangle\, \delta_{ij}$. It is clear that $(\tilde{\Lambda}_{ij})$ is positive semi-definite in U and x_0 is a degenerate point. Therefore, we have

$$\tilde{\Lambda}_{nn1} = 0 \ \text{ at } x_0.$$

This implies

$$u_{nn1}(x_0) = u_{11}(x_0). \tag{4.5}$$

On the other hand, the matrix (u_{ij}) is also positive semi-definite in U. Hence,

$$u_{nn1}(x_0) = 0. \tag{4.6}$$

Combining (4.5) and (4.6) we get $u_{11}(x_0) = 0$. By virtue of (4.1) this is impossible. Therefore, at the degenerate point x_0 we always have $w(x_0) \neq 0$.

Next, we still use U to denote a neighborhood of x_0, since $w(x_0) \neq 0$, we may also assume $w(x) \neq 0$ for any $x \in U$. For our convenience, we denote

$$\frac{u_i(x)}{w} = a_i(x), \tag{4.7}$$

then we have $\sum_{i=1}^n a_i^2 = 1$.

Below, we follow the idea of the proof of Lemma 4.1 in [7].

Let

$$S_k(\Lambda) = S_k(\lambda_1, \cdots, \lambda_n) = \sum_{1\le i_1 \le \cdots \le i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$$

be the k-th symmetric function. By our assumption that $\mu_1 \ge c^* > n^2/4$, we have

$$S_1(\Lambda) = \sum_{i=1}^n \lambda_i = \mu_1 + w^2 - nw = \mu_1 + (w - \tfrac{n}{2})^2 - \frac{n^2}{4} \ge \mu_1 - \frac{n^2}{4} > 0.$$

Recall that x_0 is the "most degenerate" point in Ω, thus there exists an integer $1 \le l \le n-1$ and a positive constant $C_0 > 0$ such that $S_l(\Lambda) \ge C_0$ for all $x \in U$ and $S_{l+1}(\Lambda(x_0)) = 0$.

We will denote $\phi(x) := S_{l+1}(\Lambda(x))$. For two functions defined in an open set $U \subset \Omega$, $x \in U$, we say that $h(x) \lesssim k(x)$ provided there exist positive constants c_1 and c_2 such that

$$(h-k)(x) \le c_1 |\nabla \phi(x)| + c_2 \phi(x).$$

We also write $h(x) \sim k(x)$ if $h(x) \lesssim k(x)$ and $k(x) \lesssim h(x)$. Moreover, we write $h \lesssim k$ if the above inequality holds in U, with constants c_1 and c_2 depending only on $\|u\|_{C^3}, n$, and C_0.

Since $S_l(\Lambda) \ge C_0 > 0$ for all $x \in U$, there is a positive constant $C > 0$ depending only on $\|u\|_{C^3}, n$, and C_0, such that

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l \ge C.$$

Let $G = \{1, 2, \cdots, l\}$ and $B = \{l+1, \cdots, n\}$ be the "good " and "bad" sets of indices, respectively. Denote $S_k(G) = S_k(\lambda_1, \cdots, \lambda_l)$, as $\phi = S_{l+1}(\Lambda)$ we have

$$0 \sim \phi(x) \sim S_l(G) \sum_{i\in B} \Lambda_{ii} \sim \sum_{i \in B} \Lambda_{ii}. \tag{4.8}$$

Recall that $\Lambda_{ii} \ge 0$, from (4.8) we get when $i \in B$, $\Lambda_{ii}(x) \sim 0$ for all $x \in U$. Therefore, for $i \in B$

$$u_{ii} \sim w. \tag{4.9}$$

Similarly, since $\phi_\alpha = \sum_{i,j} S_{l+1}^{ij} \Lambda_{ij\alpha}$ for $S_{l+1}^{ij} = \frac{\partial S_{l+1}}{\partial \Lambda_{ij}} = S_l(\Lambda|i)$, we obtain

$$0 \sim \phi_\alpha \sim S_l(G) \sum_{i \in B} \Lambda_{ii\alpha} \sim \sum_{i\in B} \Lambda_{ii\alpha}. \tag{4.10}$$

Here and below we will use $(\Lambda|i)$ to denote the matrix obtained by excluding the ith-column and ith-row from Λ. This implies

$$\sum_{i\in B} u_{ii\alpha} \sim (n-l) w_\alpha. \tag{4.11}$$

Finally, we will compute $\triangle \phi(x)$ in U. Following the steps on page 567 of [7], with [7, (4.15)] we get

$$\begin{aligned} \triangle \phi &\sim \sum_{\alpha, i} S_{l+1}^{ii} \Lambda_{ii\alpha\alpha} - 2 \sum_\alpha \sum_{i \in B, j\in G} S_{l-1}(G|j) \Lambda_{ij\alpha}^2 - S_{l-1}(G) \sum_\alpha \sum_{i,j\in B} \Lambda_{ij\alpha}^2 \\ &\sim S_l(G) \sum_\alpha \sum_{i\in B} \Lambda_{ii\alpha\alpha} - 2 S_l(G) \sum_\alpha \sum_{i\in B, j \in G} \frac{\Lambda_{ij\alpha}^2}{\lambda_j} - S_{l-1}(G) \sum_\alpha \sum_{i,j\in B} \Lambda_{ij\alpha}^2. \end{aligned} \tag{4.12}$$

A straightforward calculation with the definition of a_i in (4.7) and the assumption that (u_{ij}) is diagonalized at x_0 yields,

$$w_i = \sum_j \frac{u_j u_{ji}}{w} = a_i u_{ii}, \tag{4.13}$$

and

$$\begin{aligned} w_{ii} &= \sum_j \frac{u_{ji}^2 + u_j u_{jii}}{w} - \sum_j \frac{u_j u_{ji}}{w^2} w_i \\ &= \frac{u_{ii}^2}{w} + \sum_{j \ne i} a_j (u_{iij} - u_j) + a_i u_{iii} - \frac{a_i^2 u_{ii}^2}{w}. \end{aligned} \tag{4.14}$$

Here, we used $u_{jii} = u_{iij} - u_j$ for $i \neq j$ in $\mathbb{H}^n$. This gives,

$$\triangle w = \sum_i \frac{(1-a_i^2)u_{ii}^2}{w} - (n-1)w + \sum_i a_i(2ww_i). \tag{4.15}$$

Now for fixed $i \in B$, applying the commutation formula $u_{ii\alpha\alpha} = u_{\alpha\alpha ii} - 2u_{ii} + 2u_{\alpha\alpha}$ we obtain

$$\begin{aligned}\sum_\alpha u_{ii\alpha\alpha} &= \sum_\alpha (u_{\alpha\alpha ii} - 2u_{ii} + 2u_{\alpha\alpha})\\ &\sim (\triangle u)_{ii} - 2nw + 2\triangle u\\ &= (2w_i^2 + 2ww_{ii}) - 2nw + 2(\mu_1 + w^2)\\ &= 2a_i^2w^2 + 2w\left(\frac{u_{ii}^2}{w} + \sum_\alpha a_\alpha u_{ii\alpha} - w\right) - 2nw + 2(\mu_1 + w^2)\\ &= 2a_i^2w^2 + 2w\sum_\alpha a_\alpha u_{ii\alpha} - 2nw + 2(\mu_1 + w^2).\end{aligned} \tag{4.16}$$

Here, we have used the equations (4.9), (4.13), and (4.14). Combining (4.16) with (4.15) we get, for any fixed $i \in B$

$$\sum_\alpha \Lambda_{ii\alpha\alpha} \sim 2a_i^2w^2 + 2w\sum_\alpha a_\alpha \Lambda_{ii\alpha} + 2\mu_1 + 2w^2 - (n+1)w - \sum_\alpha \frac{(1-a_\alpha^2)u_{\alpha\alpha}^2}{w}. \tag{4.17}$$

Notice that the last term of (4.17) can be written as follows.

$$\begin{aligned}\sum_\alpha \frac{(1-a_\alpha^2)u_{\alpha\alpha}^2}{w} &= \sum_\alpha \frac{(1-a_\alpha^2)(\lambda_\alpha^2 + 2\lambda_\alpha w + w^2)}{w}\\ &= \sum_\alpha \frac{(1-a_\alpha^2)\lambda_\alpha^2}{w} + 2\sum_\alpha (1-a_\alpha^2)\lambda_\alpha + \sum_\alpha (1-a_\alpha^2)w\\ &\sim \sum_{\alpha\in G} \frac{(1-a_\alpha^2)\lambda_\alpha^2}{w} + 2\sum_\alpha \lambda_\alpha - 2\sum_\alpha a_\alpha^2\lambda_\alpha + (n-1)w.\end{aligned}$$

In view of the equality $\sum_\alpha \lambda_\alpha = \mu_1 + w^2 - nw$, equation (4.17) becomes

$$\sum_\alpha \Lambda_{ii\alpha\alpha} \lesssim a_i^2\left(2w^2 - \sum_{\alpha\in G}\frac{\lambda_\alpha^2}{w}\right) + 2w\sum_\alpha a_\alpha\Lambda_{ii\alpha} + 2\sum_\alpha a_\alpha^2\lambda_\alpha,\ i \in B, \tag{4.18}$$

where we have used $1 - a_\alpha^2 \geq a_i^2$ for any $\alpha \in G$. Therefore, in conjunction with (4.10) we conclude,

$$S_l(G)\sum_\alpha\sum_{i\in B}\Lambda_{ii\alpha\alpha} \lesssim S_l(G)\left[\sum_{i\in B} a_i^2\left(2w^2 - \sum_{\alpha\in G}\frac{\lambda_\alpha^2}{w}\right) + 2(n-l)\sum_{\alpha\in G} a_\alpha^2\lambda_\alpha\right]. \tag{4.19}$$

Now, for any fixed $j \in G$ by the Cauchy-Schwarz inequality we have

$$\sum_{i\in B}\Lambda_{iji}^2 \geq \frac{\left(\sum_{i\in B}\Lambda_{iji}\right)^2}{n-l}.$$

Using (4.11) we get

$$\begin{aligned}\sum_{i\in B}\Lambda_{iji} = \sum_{i\in B} u_{iji} &= \sum_{i\in B}(u_{iij} - u_j)\\ &\sim (n-l)(w_j - u_j) = (n-l)a_j\lambda_j.\end{aligned}$$

Therefore, we obtain for any fixed $j \in G$,

$$\sum_{i\in B}\frac{\Lambda_{iji}^2}{\lambda_j} \gtrsim (n-l)a_j^2\lambda_j. \tag{4.20}$$

Similarly, we can compute

$$\sum_{i\in B, j\in G}\frac{\Lambda_{ijj}^2}{\lambda_j} \geq \sum_{i\in B, j\in G}\frac{\Lambda_{ijj}^2}{\lambda_1} \geq \frac{1}{\lambda_1 l}\sum_{i\in B}\left(\sum_{j\in G}\Lambda_{ijj}\right)^2.$$

For fixed $i \in B$, it is easy to see that

$$\sum_{j\in G} \Lambda_{ijj} = \sum_{j\in G} u_{ijj} = \sum_{j\in G} (u_{jji} - u_i).$$

In view of equation (4.1) we get

$$\sum_{j\in G} u_{jji} + \sum_{j\in B} u_{jji} = 2ww_i.$$

Applying (4.11) we conclude for any fixed $i \in B$

$$\sum_{j\in G} u_{jji} \sim (2w - n + l)a_i w.$$

This gives

$$\sum_{j\in G} \Lambda_{ijj} \sim (2w - n)a_i w.$$

Therefore, we obtain for any fixed $i \in B$,

$$\sum_{j\in G} \frac{\Lambda_{ijj}^2}{\lambda_j} \gtrsim \frac{(2w-n)^2 a_i^2 w^2}{\lambda_1 l}. \tag{4.21}$$

Plugging (4.19), (4.20), and (4.21) into (4.12) we have

$$\begin{aligned} \triangle\phi \lesssim\ & S_l(G)\left[\sum_{i\in B} a_i^2 \left(2w^2 - \sum_{\alpha\in G}\frac{\lambda_\alpha^2}{w}\right) + 2(n-l)\sum_{\alpha\in G} a_\alpha^2 \lambda_\alpha\right] \\ & - 2S_l(G)\left[\sum_{\alpha\in G}(n-l)a_\alpha^2\lambda_\alpha + \frac{(2w-n)^2}{\lambda_1 l}\sum_{i\in B} a_i^2 w^2\right] \\ =\ & S_l(G)\sum_{i\in B} a_i^2 \left[2w^2 - \sum_{\alpha\in G}\frac{\lambda_\alpha^2}{w} - 2\frac{(2w-n)^2 w^2}{\lambda_1 l}\right]. \end{aligned} \tag{4.22}$$

Since

$$\sum_{\alpha\in G} \lambda_\alpha^2 > \frac{\left(\sum_{\alpha\in G}\lambda_\alpha\right)^2}{l} \gtrsim \frac{(\mu_1 + w^2 - nw)^2}{l},$$

it is easy to see that there exists $c^* = c^*(n) > 0$ only depending on n (e.g. $10n^2$) such that when $\mu_1 \ge c^*$ we always have,

$$\left[2w^2 - \sum_{\alpha\in G}\frac{\lambda_\alpha^2}{w} - 2\frac{(2w-n)^2 w^2}{\lambda_1 l}\right] \le 0.$$

By the strong maximum principle we conclude that $\phi \equiv 0$ in U. Repeating the above argument, we would conclude that $\phi = 0$ whenever $|\nabla u| \neq 0$ in Ω, which is impossible. $\square$

From the above discussions, we know that for any $t > 0$, the first eigenfunction of Ω_t is strictly 1-log-concave. Therefore, by continuity, we know that the first eigenfunction of Ω is 1-log-concave, that is, super log-concave. This proves Theorem 1.1.

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