

A Neumann Type Maximum Principle for the Laplace Operator on Compact Riemannian Manifolds

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Abstract

In this paper we present a proof of a Neumann type maximum principle for the Laplace operator on compact Riemannian manifolds. A key point is the simple geometric nature of the constant in the a priori estimate of this maximum principle. In particular, this maximum principle can be applied to manifolds with Ricci curvature bounded from below and diameter bounded from above to yield a maximum estimate without dependence on a positive lower bound for the volume.

1 Introduction

The main purpose of this paper is to present a proof of a Neumann type maximum principle for the Laplace operator on a closed Riemannian manifold. As a key feature of this maximum principle, the constant in the maximum estimate depends on the Riemannian manifold only in terms of the dimension and the volume-normalized Neumann isoperimetric constant. This allows us to apply it to manifolds with Ricci curvature bounded from below and diameter bounded from above to obtain a maximum principle without dependence on a positive lower bound for the volume. A special case of this maximum principle, namely Theorem C with $\Phi = 0$ has been believed to be true and used in [P] for establishing an eigenvalue pinching theorem for manifolds with positive Ricci curvature. (The accounts in [P] also suggest a belief in a general version.) But we cannot find any other reference for this maximum principle (the special case or the general case) in the literature. A corresponding maximum principle (in various formulations) for the Dirichlet boundary value problem on a domain is well-known. But its usual proof, which is an application of Moser iteration based on the Sobolev inequality, is not suitable for the Neumann type problem of

this paper for a number of reasons. In particular, the key independence from volume lower bound mentioned above requires new arguments for our Neumann type problem. Another obvious difference is that no average of the subsolution appears in the maximum principle for the Dirichlet boundary value problem, in contrast to the situation of this paper.

Consider a closed Riemannian manifold (M, g) of dimension n , where g denotes the metric. Let $L^p(M)$ denote the L^p space of functions on M , $L^p(TM)$ the L^p space of vector fields on M , and $W^{k,p}(M)$ the $W^{k,p}$ Sobolev space of functions on M . The L^p norm with respect to g will be denoted by $\|\cdot\|_p$, i.e.

$$\|f\|_p = \left(\int_M |f|^p \right)^{\frac{1}{p}}, \|\Phi\|_p = \left(\int_M |\Phi|^p \right)^{\frac{1}{p}} \quad (1.1)$$

for $f \in L^p(M)$ and $\Phi \in L^p(TM)$. (The notation of the volume form of g is often omitted in this paper.) The following volume-normalized L^p norm $\|\cdot\|_p^*$ will play an important role in this paper:

$$\|f\|_p^* = \left(\frac{1}{\text{vol}_g(M)} \int_M |f|^p \right)^{\frac{1}{p}}, \|\Phi\|_p^* = \left(\frac{1}{\text{vol}_g(M)} \int_M |\Phi|^p \right)^{\frac{1}{p}}, \quad (1.2)$$

where $\text{vol}_g(M)$ denotes the volume of (M, g) .

The average of a function $u \in L^1(M)$ on M will be denoted by u_M , i.e.

$$u_M = \frac{1}{\text{vol}_g(M)} \int_M u. \quad (1.3)$$

For a function u on M we denote its positive part by u^+ and its negative part by u^- , i.e. $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. The Laplace operator Δ is the negative Laplacian, i.e. $\Delta u = \text{div} \nabla u$. Let $C_{I,N}^*(M, g)$ denote the volume-normalized Neumann isoperimetric constant, which is defined in terms of the Neumann isoperimetric constant $C_{I,N}(M, g)$, see Section 2.

Theorem A *Assume $n \geq 3$. Let u be a function in $W^{1,\alpha}(M)$ with $\alpha > n$, which satisfies*

$$\Delta u \geq f + \text{div } \Phi \quad (1.4)$$

in the weak sense for a measurable function f on M such that $f^- \in L^p(M)$ and a vector field $\Phi \in L^{2p}(TM)$ with $p > \frac{n}{2}$, i.e.

$$\int_M \nabla u \cdot \nabla \phi \leq - \int_M f \phi + \int_M \Phi \cdot \nabla \phi \quad (1.5)$$

for all nonnegative $\phi \in W^{1,2}(M)$ (equivalently, all nonnegative $\phi \in W^{1,\frac{\alpha}{\alpha-1}}(M)$). Then we have

$$\sup_M u \leq u_M + C(n, p, C_{N,I}^*(M, g))(\|f^-\|_p^* + \|\Phi\|_{2p}^*) \quad (1.6)$$

with a positive constant $C(n, p, C_{N,I}^*(M, g))$ depending only on n, p and $C_{N,I}^*(M, g)$. This constant depends continuously and increasingly on $C_{N,I}^*(M, g)$.

The classical strong maximum principle says that $u \equiv u_M$ if $\Delta u \geq 0$ (in the weak sense). Theorem A includes this as a special corollary. But the main point of Theorem A lies in the quantitative estimate (1.6) and the simple geometric nature of the constant $C(n, p, C_{N,I}^*(M, g))$ in the estimate. As emphasized above, no other data from the metric g such as the volume are involved in this constant.

In contrast to traditional estimates of the maximum principle type, the estimate (1.6) is not scaling invariant. In other words, the estimate obtained with respect to a rescaled metric and the corresponding rescaled f and Φ differs from the original estimate. This non-invariance is brought into the estimate by a construction in the proof of Lemma 4.2, see Remark 3 in Section 4. (For a discussion of the behavior of the estimate under rescaling of the metric see Remark 4.) Without breaking the scaling invariance it would be impossible to obtain a maximum estimate in which the constant depends solely on the dimension n , the exponent p and the volume-normalized Neumann isoperimetric constant. This is one of the key features of our arguments. (Scaling invariant maximum estimates can also be derived, see [Y].)

As a consequence of Theorem A and S. Gallot's estimate of the volume-normalized Neumann isoperimetric constant in [Ga1] (see Theorem 2.10) we obtain the following result which involves a lower bound for the Ricci curvature and an upper bound for the diameter. For convenience, we define the *diameter rescaled Ricci curvature* of a unit tangent vector v to be $\hat{Ric}(v, v) = \text{diam}_g(M)^2 Ric(v, v)$, where $\text{diam}_g(M)$ denotes the diameter of (M, g) . We set $\kappa_{\hat{Ric}} = \min_{v \in TM, |v|=1} \hat{Ric}(v, v)$ and $\hat{\kappa}_{\hat{Ric}} = |\kappa_{\hat{Ric}}^-| = |\min\{\kappa_{\hat{Ric}}, 0\}|$.

Theorem B Assume $n \geq 3$. Let u be a function in $W^{1,\alpha}(M)$ with $\alpha > n$ satisfying

$$\Delta u \geq f + \text{div } \Phi \quad (1.7)$$

in the weak sense for a measurable function f such that $f^- \in L^p(M)$ and a vector field $\Phi \in L^{2p}(TM)$ with $p > \frac{n}{2}$. Then we have

$$\sup_M u \leq u_M + C(n, p, \hat{\kappa}_{\hat{Ric}}, \text{diam}_g(M))(\|f^-\|_p^* + \|\Phi\|_{2p}^*). \quad (1.8)$$

with a positive constant $C(n, p, \hat{\kappa}_{\hat{Ric}}, \text{diam}_g(M))$ depending only on $n, p, \hat{\kappa}_{\hat{Ric}}$ and the diameter. This constant depends continuously on its arguments and increasingly on

$\hat{\kappa}_{\hat{Ric}}$ and $\text{diam}_g(M)$.

If we assume an upper bound D for the diameter and a nonpositive lower bound κ for the Ricci curvature, then we have $\min_M \hat{Ric} \geq D^2 \kappa$ and $C(n, p, \hat{\kappa}_{\hat{Ric}}, \text{diam}_g(M)) \leq C(n, p, D^2 |\kappa|, D)$. Hence the estimates (1.8) can be applied. We state a corollary for the case of positive Ricci curvature. We formulate it under the assumption $Ric \geq n - 1$, which can always be achieved by rescaling.

Theorem C *Assume $n \geq 3$ and that the Ricci curvature satisfies $Ric \geq n - 1$. Let u be a function in $W^{1,\alpha}(M)$ with $\alpha > n$ satisfying*

$$\Delta u \geq f + \text{div } \Phi \quad (1.9)$$

in the weak sense for a measurable function f on M such that $f^- \in L^p(M)$ and a vector field $\Phi \in L^{2p}(TM)$ with $p > \frac{n}{2}$. Then we have

$$\sup_M u \leq u_M + C(n, p)(\|f\|_p^* + \|\Phi\|_{2p}^*) \quad (1.10)$$

with a positive constant $C(n, p)$ depending only on n and p .

Analogous results hold true if we assume an upper bound for the diameter, and a lower bound for the Ricci curvature in a suitable integral sense, thanks Gallot's and Petersen-Sprouse's estimates for the Neumann isoperimetric constant in [Ga2] and [PS]. We omit the obvious statements of those results.

Remark 1 In the above results we restrict to dimensions $n \geq 3$. The 2-dimensional analogues also hold true, see [Y]. We would also like to mention that it is straightforward to extend the above results to compact manifolds with boundary under the Neumann boundary condition. One can also extend the above results to general elliptic operators of divergence form, see [Y].

Remark 2 Theorem A is also valid if we replace in the definition of $C_{I,N}^*(M, g)$ the Neumann isoperimetric constant $C_{N,I}(M, g)$ by the Poincaré-Sobolev constant $C_{P,S}(M, g)$ (see Section 2 for its definition). Indeed, it is the Poincaré inequality (2.3), the Poincaré-Sobolev inequality (2.4) and the Sobolev inequality (2.5) which are employed in our arguments. The Neumann isoperimetric constant appears in these inequalities. Obviously, the Poincaré-Sobolev inequality (2.4) can be reformulated in terms of the Poincaré-Sobolev constant. Then the Poincaré inequality (2.3) and the Sobolev inequality (2.5) follow as corollaries, with the constants suitably modified. We formulate Theorem A in terms of the Neumann isoperimetric constant because we consider it to be a more fundamental quantity.

The proof of Theorem A involves several ingredients. One is Moser iteration based on the Sobolev inequality. Various versions of this technique have been used in many situations, but the way it is done in this paper is new, see the proof of Lemma 4.2. It is in this proof that the scaling invariance is broken, as mentioned above. On the other hand, from this proof one can see that the technique of Moser iteration alone cannot lead to a maximum estimate for $u - u_M$ in terms of f and Φ . Instead, the estimate one obtains also depends on the L^2 norm of $(u - u_M)^+$. Without using additional tools it seems impossible to go any further. Our strategy for overcoming this difficulty is to employ the Green function G_0 of the Laplace operator. First we combine Lemma 4.2 with the Poincaré inequality to establish Theorem 4.3 which is the corresponding maximum principle for solutions (rather than subsolutions). Using this result we reduce the right hand side of (1.4) to a constant. Then we utilize the Green function G_0 to obtain the desired estimate. Employing the Green function is crucial for the whole scheme.

There is an additional subtlety here. Usually, maximum principles based on Moser iteration hold true for all subsolutions u in the Sobolev space $W^{1,2}(M)$. (This is the case in Lemma 4.2 (for subsolutions) and Theorem 4.3 (for solutions).) In the situation of Theorem A (hence also Theorem B and Theorem C), we have to require $u \in W^{1,\alpha}(M)$ for $\alpha > n$. This restriction stems from the involvement of the Green function. Using additional tools, one can extend Theorem A to $u \in W^{1,2}(M)$, provided that $\Phi = 0$, see [Y]. It remains open whether one can extend the full Theorem A to $u \in W^{1,2}(M)$. (See also [Y] for a weaker maximum principle which holds true for all $u \in W^{1,2}(M)$.)

In the above scheme of utilizing the Green function G_0 , a lower bound for G_0 is needed. In [Si], a lower bound for G_0 in terms of $\hat{\kappa}_{Ric}$, the volume and the diameter is obtained. This lower bound is sufficient for establishing the estimate (1.8) in Theorem B and the estimate (1.10) in Theorem C, but is not suitable for establishing the general estimate (1.6) in Theorem A. Following the arguments in [CL] and [Si], we derive in Section 3 a lower bound for G_0 which is proportional to $vol_g(M)^{-1}$ with a factor given in terms of the volume-normalized Neumann isoperimetric constant. This form of lower bound is exactly what we need for establishing Theorem A. It is also of independent interest.

We would like to mention that Theorem C is sufficient for the purpose of [P] because all involved functions in [P] are at least Lipschitz continuous. We would also like to mention that Theorem A (or Theorem 4.2) leads to an estimate for the L^p norm of the Green function G_0 for each $0 < p < \frac{n}{n-2}$ (thanks an observation by Xiaodong Wang) and an estimate for the L^q norm of the gradient of G_0 for each $0 < q < \frac{n}{n-1}$. This will be presented elsewhere.

We would like to thank Xiaodong Wang for bringing the question regarding the validity of Theorem C (with $\Phi = 0$) to our attention. The first named author would also like to acknowledge many helpful discussions with Xiaodong Wang, and also with Jian Song.

2 The Neumann Isoperimetric Constant

Consider a closed Riemannian manifold (M, g) of dimension n . The Neumann isoperimetric constant of (M, g) is defined to be

$$C_{N,I}(M, g) = \sup\left\{\frac{\text{vol}(\Omega)^{\frac{n-1}{n}}}{A(\partial\Omega)} : \Omega \subset M \text{ is a } C^1 \text{ domain, } \text{vol}_g(\Omega) \leq \frac{1}{2}\text{vol}_g(M)\right\}. \quad (2.1)$$

The Poincaré-Sobolev constant (for the exponent 2) of (M, g) is defined to be

$$C_{P,S}(M, g) = \sup\{\|u - u_M\|_{\frac{2n}{n-2}} : u \in C^1(M), \|\nabla u\|_2 = 1\}. \quad (2.2)$$

We have the following Poincaré inequality, Poincaré-Sobolev inequality and Sobolev inequality. See [Y] for their proofs. (For these inequalities with somewhat different constants see [Si].) The Poincaré-Sobolev inequality (2.4) gives an upper bound of the Poincaré-Sobolev constant in terms of the Neumann isoperimetric constant.

Lemma 2.1 *There hold for all $u \in W^{1,2}(M)$*

$$\|u - u_M\|_2 \leq \frac{2(n-1)}{n-2} C_{N,I}(M, g) \text{vol}_g(M)^{\frac{1}{n}} \|\nabla u\|_2, \quad (2.3)$$

$$\|u - u_M\|_{\frac{2n}{n-2}} \leq \frac{4(n-1)}{n-2} C_{N,I}(M, g) \|\nabla u\|_2, \quad (2.4)$$

and

$$\|u\|_{\frac{2n}{n-2}} \leq \frac{2(n-1)}{n-2} C_{N,I}(M, g) \|\nabla u\|_2 + \frac{\sqrt{2}}{\text{vol}_g(M)^{\frac{1}{n}}} \|u\|_2, \quad (2.5)$$

whenever $n \geq 3$.

It is convenient to use the following volume-normalized Neumann isoperimetric constant:

$$C_{I,N}^*(M, g) = C_{N,I}(M, g) \text{vol}_g(M)^{\frac{1}{n}}, \quad (2.6)$$

which was first introduced by J. Cheeger in his study of the first eigenvalue of the Laplace operator [Che]. Note that $C_{I,N}^*(M, g)$ is scaling invariant, while $C_{I,M}^*(M, g)$ has the same scaling weight as the n -th root of the volume, or the diameter. In terms of $C_{I,N}^*(M, g)$ and the volume-normalized L^p norms, Lemma 2.1 can be reformulated as follows.

Lemma 2.2 *There hold for all $u \in W^{1,2}(M)$*

$$\|u - u_M\|_2^* \leq \frac{2(n-1)}{n-2} C_{N,I}^*(M, g) \|\nabla u\|_2^*, \quad (2.7)$$

$$\|u - u_M\|_{\frac{2n}{n-2}}^* \leq \frac{4(n-1)}{n-2} C_{N,I}^*(M, g) \|\nabla u\|_2^*, \quad (2.8)$$

and

$$\|u\|_{\frac{2n}{n-2}}^* \leq \frac{2(n-1)}{n-2} C_{N,I}^*(M, g) \|\nabla u\|_2^* + \sqrt{2} \|u\|_2^*, \quad (2.9)$$

whenever $n \geq 3$.

The following estimate of the volume-normalized Neumann isoperimetric constant easily follows from S. Gallot's corresponding estimate in [Gal].

Theorem 2.3 *There holds*

$$C_{N,I}^*(g, M) \leq C(n, \hat{\kappa}_{\hat{Ric}}) \text{diam}_g(M), \quad (2.10)$$

where $C(n, \hat{\kappa}_{\hat{Ric}})$ is a positive constant depending only on n and $\hat{\kappa}_{\hat{Ric}}$. It depends continuously and increasingly on $\hat{\kappa}_{\hat{Ric}}$.

Proof. We rescale to make the diameter equal one. Then we apply the estimate for the Neumann isoperimetric constant in [Gal]. Expressing the estimate in terms of the original metric we arrive at (2.10). ■

3 The Green Function

Consider a closed Riemannian manifold (M, g) of dimension n as before. Let $G_0(x, y)$ be the unique Green function of the Laplace operator Δ such that $\int_M G_0(x, y) dy = 0$ for all $x \in M$, where dy denotes the volume form of g . Thus we have

$$u(x) = u_M - \int_M G_0(x, y) \Delta_y u(y) dy \quad (3.1)$$

for all $u \in C^\infty(M)$, where Δ_y means Δ with the subscript indicating the argument. (Similar notations will be used below.) $G_0(x, y)$ is smooth away from $x = y$. Moreover, $G_0(x, y) = G_0(y, x)$ for all $x, y \in M, x \neq y$. In this section we present some basic facts about G_0 and derive a lower bound of G_0 in terms of $C_{N,I}(M, g)$ and the volume.

Lemma 3.1 Assume $n \geq 3$. Then there holds

$$G_0(x, \cdot) \in W^{1,\beta}(M) \quad (3.2)$$

for all $0 < \beta < \frac{n}{n-1}$.

Proof. By e.g. [Theorem 4.17, A] we have $|G_0(x, y)| \leq Cd(x, y)^{2-n}$ and $|\nabla_y G_0(x, y)| \leq Cd(x, y)^{1-n}$ for all $x, y \in M, x \neq y$ and a positive constant C depending on g . Consequently, we have

$$G_0(x, \cdot) \in L^{q_1}(M), \nabla_y G_0(x, y) \in L^{q_2}(M) \quad (3.3)$$

for all $0 < q_1 < \frac{n}{n-2}$ and all $0 < q_2 < \frac{n}{n-1}$. Then it follows easily that $G(x, \cdot) \in W^{1,p}(M)$ for all $x \in M$ and $0 < p < \frac{n}{n-1}$. Indeed, we have for an arbitrary $x \in M$ and small $\epsilon > 0$

$$\begin{aligned} \int_{M-B_\epsilon(x)} G_0(x, y) \operatorname{div}_y \Phi(y) dy &= - \int_{M-B_\epsilon(x)} \nabla_y G_0(x, y) \cdot \Phi(y) dy \\ &\quad + \int_{\partial B_\epsilon(x)} G_0(x, y) \Phi(y) \cdot \nu(y) dy \end{aligned} \quad (3.4)$$

for all smooth vector fields Φ on M , where ν denotes the inward unit normal of the geodesic sphere $\partial B_\epsilon(x)$. Since $|G_0(x, y)| \leq C\epsilon^{2-n}$ on $\partial B_\epsilon(x)$ we can let $\epsilon \rightarrow 0$ to arrive at

$$\int_M G_0(x, y) \operatorname{div}_y \Phi(y) dy = - \int_M \nabla_y G_0(x, y) \cdot \Phi(y) dy. \quad (3.5)$$

By (3.3) and (3.5) we infer that $G(x, \cdot) \in W^{1,q}(M)$ for all $0 < q < \frac{n}{n-1}(M)$ and that (3.5) holds true for all $\Phi \in W^{1,p}(TM)$ whenever $p > n$, where $W^{1,p}(TM)$ denotes the $W^{1,p}$ Sobolev space of vector fields on M . ■

Lemma 3.2 Let $u \in W^{1,q}(M)$ with $q > n$. Then

$$u(x) = u_M + \int_M \nabla_y G_0(x, y) \cdot \nabla_y u(y) dy \quad (3.6)$$

holds true for a.e. $x \in M$.

Proof. By Lemma 3.1, we can integrate (3.1) by parts to deduce (3.6) for all $u \in C^\infty(M)$. Applying Lemma 3.1 and a limiting argument we then conclude that (3.6) holds true for each $u \in W^{1,q}(M)$ a.e. as long as $q > n$.

■

Next let $H(x, y, t)$ be the heat kernel for Δ , i.e.

$$\frac{\partial}{\partial t} H(x, y, t) = \Delta_y H(x, y, t) \quad (3.7)$$

for $t > 0$ and

$$\lim_{t \rightarrow 0} H(x, y, t) = \delta_x \quad (3.8)$$

in the sense of distributions, where δ_x is the Dirac δ -function with center x . H is symmetric in x, y and smooth away from $x = y, t = 0$. We have the basic representation formula

$$u(x, t) = \int_0^t d\tau \int_M H(x, y, t - \tau) \left(\frac{\partial}{\partial \tau} - \Delta_y \right) u(y, \tau) dy + \int_M H(x, y, t) u(y, 0) dy \quad (3.9)$$

for all smooth functions u and $t > 0$. Note that $H(x, y, t) > 0$ for $t > 0$ and all $x, y \in M$. We set

$$G(x, y, t) = H(x, y, t) - \frac{1}{\text{vol}_g(M)}. \quad (3.10)$$

Choosing $u(x, t) \equiv 1$ in (3.9) we deduce

$$\int_M H(x, y, t) dy = 1 \quad (3.11)$$

and hence

$$\int_M G(x, y, t) dy = 0 \quad (3.12)$$

for all $x \in M$ and $t > 0$.

Lemma 3.3 *Assume $n \geq 3$. Then there holds*

$$G_0(x, y) = \int_0^\infty G(x, y, t) dt. \quad (3.13)$$

Proof. We have

$$\left| H(x, y, t) - \frac{1}{\text{vol}_g(M)} \right| \leq C t^{-\frac{n}{2}} \quad (3.14)$$

for a certain positive constant C depending on g (for a geometric estimate of C see [CL]). On the other hand, we have the inequality (see e.g. [Proposition VII.3.5, Ch])

$$|\frac{H(x, y, t)}{\mathcal{H}(x, y, t)} - 1| \leq Cd(x, y) \quad (3.15)$$

for all $t > 0$ and $x, y \in M$ with $d(x, y) \leq \frac{1}{4}inj_g(M)$, where C is a positive constant depending on g , $inj_g(M)$ denotes the injectivity radius of (M, g) , and

$$\mathcal{H}(x, y, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{d(x, y)^2}{4t}}. \quad (3.16)$$

By (3.9) we have for a smooth function $u(x)$

$$\begin{aligned} u(x) &= - \int_0^t d\tau \int_M H(x, y, t - \tau) \Delta_y u(y) dy + \int_M H(x, y, t) u(y) dy \\ &= - \int_0^t d\tau \int_M G(x, y, t - \tau) \Delta_y u(y) dy + \int_M G(x, y, t) u(y) dy + u_M \\ &= - \int_0^t ds \int_M G(x, y, s) \Delta_y u(y) dy + \int_M G(x, y, t) u(y) dy + u_M \end{aligned} \quad (3.17)$$

By (3.14) and (3.15) we can let $t \rightarrow \infty$ in (3.17) to obtain

$$u(x) = - \int_M (\int_0^\infty G(x, y, s) ds) \Delta_y u(y) dy + u_M. \quad (3.18)$$

On the other hand, by (3.12) we deduce

$$\int_M (\int_0^\infty G(x, y, s) ds) dy = 0 \quad (3.19)$$

for all $x \in M$. We conclude that (3.13) holds true. ■

Lemma 3.4 *There holds*

$$G(x, y, t + s) = \int_M G(x, z, s) G(z, y, t) dz \quad (3.20)$$

for all $x, y \in M$ and $t > 0, s > 0$, where dz denotes the volume form of g with $z \in M$ as the argument. In particular, we have

$$G(x, x, t) = \int_M G(x, y, \frac{t}{2})^2 dy \quad (3.21)$$

and it follows that $G(x, x, t) > 0$ for all $x \in M$ and $t > 0$.

Proof. Note that $G(x, y, t)$ satisfies the heat equation

$$\frac{\partial}{\partial t}G(x, y, t) = \Delta_y G(x, y, t) = \Delta_x G(x, y, t). \quad (3.22)$$

Choosing $u(x, t) = G(x, y, t + s)$ in (3.9) for each fixed y we deduce, on account of (3.22) and (3.12)

$$G(x, y, t + s) = \int_M H(x, z, t)G(z, y, s)dz = \int_M G(x, z, t)G(z, y, s)dz. \quad (3.23)$$

Switching t with s we arrive at the desired equation (3.20).

The formula (3.21) follows immediately and hence $G(x, x, t) \geq 0$. If $G(x, x, t_0) = 0$ for some x and $t_0 > 0$, then (3.21) implies that $G(x, y, \frac{t_0}{2}) = 0$ for all $y \in M$. Then $G(x, y, t) = 0$ for all $y \in M$ and $t \geq \frac{t_0}{2}$, because $G(x, y, t)$ satisfies the heat equation. It follow that $H(x, y, t) = \text{vol}_g(M)^{-1}$ for all $y \in M$ and $t \geq \frac{t_0}{2}$. This contradicts (3.9) as is easy to see. We conclude that $G(x, x, t) > 0$ for all $x \in M$ and $t > 0$. ■

Theorem 3.5 *Assume $n \geq 3$. Then there holds*

$$G_0(x, y) \geq -C_0(n)C_{I,N}^*(M, g)^2 \text{vol}_g(M)^{-1} \quad (3.24)$$

for all $x, y \in M, x \neq y$, where

$$C_0(n) = \frac{8n^2(n-1)^2}{(n-2)^3} \left(\frac{n-2}{2} \right)^{\frac{4}{n}}. \quad (3.25)$$

Proof. This follows from the arguments in [CL] and [Si] with some modification. By the rescaling invariance of (3.24) we can assume $\text{vol}_g(M) = 1$. Differentiating the equation (3.20), setting $y = x$ and replacing t and s by $\frac{t}{2}$ we deduce for $t > 0$

$$\frac{\partial G}{\partial t}(x, x, t) = \int_M \frac{\partial G}{\partial t}(x, z, \frac{t}{2})G(x, z, \frac{t}{2})dz = \int_M (\Delta_z G(x, z, \frac{t}{2}))G(x, z, \frac{t}{2})dz. \quad (3.26)$$

We integrate (3.26) by parts to derive

$$\frac{\partial G}{\partial t}(x, x, t) = - \int_M |\nabla_z G(x, z, \frac{t}{2})|^2 dz. \quad (3.27)$$

Applying the Poincaré-Sobolev inequality (2.4) we then obtain

$$-\frac{\partial G}{\partial t}(x, x, t) \geq \left(\frac{4(n-1)}{n-2} C_{N,I}(M, g) \right)^{-2} \left(\int_M |G(x, z, \frac{t}{2})|^{\frac{2n}{n-2}} dz \right)^{\frac{n-2}{n}}. \quad (3.28)$$

By Hölder's inequality we have

$$\begin{aligned} \left(\int_M |G(x, z, \frac{t}{2})|^2 dz \right)^{\frac{n+2}{n}} &= \left(\int_M |G(x, z, \frac{t}{2})|^{\frac{4}{n+2}} |G(x, z, \frac{t}{2})|^{\frac{2n}{n+2}} dz \right)^{\frac{n+2}{n}} \\ &\leq \left(\int_M |G(x, z, \frac{t}{2})|^{\frac{2n}{n-2}} dz \right)^{\frac{n-2}{n}} \left(\int_M |G(x, z, \frac{t}{2})| dz \right)^{\frac{4}{n}}. \end{aligned} \quad (3.29)$$

Next observe that $\int_M |G(x, z, t)| dz \leq 2$ because $H(x, z, t) > 0$. Hence we arrive at

$$-\frac{\partial G}{\partial t}(x, x, t) \geq C \left(\int_M |G(x, z, \frac{t}{2})|^2 dz \right)^{\frac{n+2}{n}} = CG(x, x, t)^{\frac{n+2}{n}}, \quad (3.30)$$

where

$$C = \left(\frac{4(n-1)}{n-2} C_{N,I}(M, g) \right)^{-2}. \quad (3.31)$$

Integrating (3.30) we derive

$$G(x, x, t)^{-\frac{2}{n}} \geq G(x, x, s)^{-\frac{2}{n}} + \frac{n}{2} C(t-s) \quad (3.32)$$

for $t > s > 0$. (Note that $G(x, x, t) > 0$ by Lemma 3.4.) Letting $s \rightarrow 0$ we infer $G(x, x, t)^{-\frac{2}{n}} \geq \frac{n}{2} Ct$, and hence $G(x, x, t) \leq C^{-\frac{n}{2}} (\frac{n}{2})^{\frac{n}{2}} t^{-\frac{n}{2}}$. Now we have by Lemma 3.4

$$\begin{aligned} |G(x, y, t)| &= \left| \int_M G(x, z, \frac{t}{2}) G(z, y, \frac{t}{2}) dz \right| \\ &\leq \left(\int_M G(x, z, \frac{t}{2})^2 dz \right)^{\frac{1}{2}} \left(\int_M G(z, y, \frac{t}{2})^2 dz \right)^{\frac{1}{2}} \\ &= G(x, x, \frac{t}{2})^{\frac{1}{2}} G(y, y, \frac{t}{2})^{\frac{1}{2}} \leq C^{-\frac{n}{2}} (\frac{n}{2})^{\frac{n}{2}} t^{-\frac{n}{2}}. \end{aligned} \quad (3.33)$$

Since $H(x, y, t) > 0$ we have $G(x, y, t) \geq -\frac{1}{\text{vol}_g(M)} = -1$. We deduce for each $\tau > 0$

$$\begin{aligned} G(x, y) &= \int_0^\infty G(x, y, t) dt \geq - \int_0^\tau dt - \int_\tau^\infty C^{-\frac{n}{2}} (\frac{n}{2})^{\frac{n}{2}} t^{-\frac{n}{2}} dt \\ &= -\tau - \frac{n-2}{2} C^{-\frac{n}{2}} (\frac{n}{2})^{\frac{n}{2}} \tau^{-\frac{n-2}{2}} = -\tau - C_1 \tau^{-\frac{n-2}{2}}, \end{aligned} \quad (3.34)$$

where $C_1 = \frac{n-2}{2} C^{-\frac{n}{2}} (\frac{n}{2})^{\frac{n}{2}}$. The maximum of the function $\tau + C_1 \tau^{-\frac{n-2}{2}}$ is achieved at $\tau = (C_1(n-2)/2)^{2/n}$ and hence equals

$$\frac{n}{n-2} (C_1 \frac{n-2}{2})^{\frac{2}{n}} = \frac{8n^2(n-1)^2}{(n-2)^3} \left(\frac{n-2}{2} \right)^{\frac{4}{n}} C_{I,N}(M, g)^2.$$

we arrive at

$$G(x, y) \geq -\frac{8n^2(n-1)^2}{(n-2)^3} \left(\frac{n-2}{2}\right)^{\frac{4}{n}} C_{I,N}(M, g)^2, \quad (3.35)$$

which leads to (3.24) by rescaling.

4 Neumann Type Maximum Principles

In this section we consider a fixed closed Riemannian manifold (M, g) of dimension $n \geq 3$ as before.

Lemma 4.1 1) *There hold for $f_1 \in L^p(M)$ and $f_2 \in L^q(M)$ with $p^{-1} + q^{-1} = 1$*

$$\|f_1 f_2\|_1^* \leq \|f_1\|_p^* \cdot \|f_2\|_q^*. \quad (4.1)$$

2) *There holds for $p \geq q \geq 1$ and $f \in L^p(M)$*

$$\|f\|_q^* \leq \|f\|_p^*. \quad (4.2)$$

Proof. These follow straightforwardly from the classical Hölder inequality. ■

Lemma 4.2 *Let $n \geq 3$. Assume that $u \in W^{1,2}(M)$ satisfies*

$$\Delta u \geq f + \operatorname{div} \Phi \quad (4.3)$$

in the weak sense, i.e.

$$\int_M \nabla u \cdot \nabla \phi \leq - \int_M f \phi + \int_M \Phi \cdot \nabla \phi \quad (4.4)$$

for all nonnegative $\phi \in W^{1,2}(M)$, where f is a measurable function on M such that $f^- \in L^p(M)$, and $\Phi \in L^{2p}(TM)$, with $p > \frac{n}{2}$. Then we have

$$\begin{aligned} \sup_M (u - \lambda) &\leq AC_1(\|f^-\|_p^* + \|\Phi\|_{2p}^*) \\ &\quad + A(C_1 + \sqrt{2})\|(u - \lambda)^+\|_2^* \end{aligned} \quad (4.5)$$

for each $\lambda \in \mathbf{R}$, where A and C_1 are positive numbers depending only on n, p and $C_{N,I}^(M, g)$. Their explicit values are given in the proof below.*

Proof. The arguments here are inspired by some arguments in [GT]. A special new feature in our argument is the use of the volume-normalized isoperimetric constant (or Sobolev constant) and the volume-normalized L^p norms. We set

$$a = \|f^-\|_p^* + \|\Phi\|_{2p}^*. \quad (4.6)$$

Then we set $b = a$ if $a > 0$ and $b = 1$ if $a = 0$. For $L > |\lambda|$ we set $w = \min\{(u - \lambda)^+, L\} + b$. Then $w \in W^{1,2}(M)$ and is bounded. It follows that $w^\gamma \in W^{1,2}(M)$ for $\gamma \geq 1$. Moreover, we have $\nabla w = 0$ if $u \geq \lambda + L$, $\nabla w = \nabla u$ if $\lambda < u < L + \lambda$, and $\nabla w = 0$ if $u \leq \lambda$. Choosing $\phi = w^\gamma(\text{Vol}M)^{-1}$ with $\text{Vol}M = \text{vol}_g(M)$ in (4.4) we obtain

$$\begin{aligned} \frac{\gamma}{\text{Vol}M} \int_M |\nabla w|^2 w^{\gamma-1} &\leq -\frac{1}{\text{Vol}M} \int_M f w^\gamma + \frac{\gamma}{\text{Vol}M} \int_M w^{\gamma-1} \Phi \cdot \nabla w \\ &\leq \frac{1}{\text{Vol}M} \int_M |f^-| w^\gamma + \frac{\gamma}{\text{Vol}M} \int_M w^{\gamma-1} \Phi \cdot \nabla w. \end{aligned} \quad (4.7)$$

First we choose $\gamma = 1$ to deduce

$$\begin{aligned} \|\nabla w\|_2^{*2} &\leq \|f^- w\|_1^* + \|\Phi \cdot \nabla w\|_1^* \\ &\leq \frac{1}{2} \|f^-\|_2^{*2} + \frac{1}{2} \|w\|_2^{*2} + \frac{1}{2} \|\Phi\|_2^{*2} + \frac{1}{2} \|\nabla w\|_2^{*2}, \end{aligned} \quad (4.8)$$

where we have used Lemma 4.1. It follows that

$$\|\nabla w\|_2^{*2} \leq \|f^-\|_2^{*2} + \|\Phi\|_{2p}^{*2} + \|w\|_2^{*2} \leq \|f^-\|_p^{*2} + \|\Phi\|_{2p}^{*2} + \|w\|_2^{*2}, \quad (4.9)$$

where the second inequality follows from Lemma 4.1. Applying the Sobolev inequality (2.9) we then deduce

$$\begin{aligned} \|w\|_{\frac{2n}{n-2}}^* &\leq \frac{2(n-1)}{n-2} C_{N,I}^*(M, g) \|\nabla w\|_2^* + \sqrt{2} \|w\|_2^* \\ &\leq C_1 (\|f^-\|_p^* + \|\Phi\|_{2p}^*) + (C_1 + \sqrt{2}) \|w\|_2^*, \end{aligned} \quad (4.10)$$

where $C_1 = \frac{2(n-1)}{n-2} C_{N,I}^*(M, g)$.

Next we consider general $\gamma \geq 1$. We deduce

$$\begin{aligned} \frac{\gamma}{\text{Vol}M} \int_M |\nabla w|^2 w^{\gamma-1} &\leq \frac{1}{b \text{Vol}(M)} \int_M |f^-| w^{\gamma+1} + \frac{\gamma}{b \text{Vol}(M)} \int_M w^\gamma |\Phi| |\nabla w| \\ &\leq \frac{1}{b} \|f^- w^{\gamma+1}\|_1^* + \frac{\gamma}{2} \|\nabla w\|_2^{*2} w^{\gamma-1} + \frac{\gamma}{2b^2} \|w^{\gamma+1} |\Phi|^2\|_1^*. \end{aligned} \quad (4.11)$$

It follows that, on account of Lemma 4.1

$$\begin{aligned} &\frac{\gamma}{2} \|\nabla w\|_2^{*2} w^{\gamma-1} \\ &\leq \frac{1}{b} \|f^- w^{\gamma+1}\|_1^* + \frac{\gamma}{2b^2} \|w^{\gamma+1} |\Phi|^2\|_1^* \\ &\leq \frac{1}{b} \|f^-\|_p^* \cdot \|w\|_{(\gamma+1)\frac{p}{p-1}}^{*(\gamma+1)} + \frac{\gamma}{2b^2} \|w\|_{(\gamma+1)\frac{p}{p-1}}^{*(\gamma+1)} \cdot \|\Phi\|_{2p}^{*2} \\ &\leq \|w\|_{(\gamma+1)\frac{p}{p-1}}^{*(\gamma+1)} + \frac{\gamma}{2} \|w\|_{(\gamma+1)\frac{p}{p-1}}^{*(\gamma+1)}. \end{aligned} \quad (4.12)$$

It follows that

$$\|\nabla w^{\frac{\gamma+1}{2}}\|_2^{*2} \leq \frac{(\gamma+2)(\gamma+1)^2}{4\gamma} \|w\|_{(\gamma+1)\frac{p}{p-1}}^{*(\gamma+1)}. \quad (4.13)$$

Now we apply the Sobolev inequality (2.9) and (4.13) to deduce

$$\begin{aligned} \|w\|_{(\gamma+1)\frac{n}{n-2}}^{*\frac{\gamma+1}{2}} &= \|w^{\frac{\gamma+1}{2}}\|_{\frac{2n}{n-2}}^{*} \leq A_\gamma \|w\|_{(\gamma+1)\frac{p}{p-1}}^{*\frac{\gamma+1}{2}} + \sqrt{2} \|w^{\frac{\gamma+1}{2}}\|_2^* \\ &= A_\gamma \|w\|_{(\gamma+1)\frac{p}{p-1}}^{*\frac{\gamma+1}{2}} + \sqrt{2} \|w\|_{\gamma+1}^{*\frac{\gamma+1}{2}} \\ &\leq (A_\gamma + \sqrt{2}) \|w\|_{(\gamma+1)\frac{p}{p-1}}^{*\frac{\gamma+1}{2}}, \end{aligned} \quad (4.14)$$

where

$$A_\gamma = C_{N,I}^*(M, g) \frac{(n-1)(\gamma+1)}{n-2} \sqrt{\frac{\gamma+2}{\gamma}}. \quad (4.15)$$

Consequently, we obtain

$$\|w\|_{(\gamma+1)\frac{n}{n-2}}^* \leq (A_\gamma + \sqrt{2})^{\frac{2}{\gamma+1}} \|w\|_{(\gamma+1)\frac{p}{p-1}}^*. \quad (4.16)$$

Replacing $\gamma+1$ by $\gamma \geq 2$ we infer

$$\|w\|_{\gamma\frac{n}{n-2}}^* \leq (A_{\gamma-1} + \sqrt{2})^{\frac{2}{\gamma}} \|w\|_{\gamma\frac{p}{p-1}}^*. \quad (4.17)$$

Now we choose $\gamma_0 = 1 + \frac{n(p-2)+2p}{(n-2)p}$ and $\gamma_k = \gamma_{k-1} \frac{n(p-1)}{(n-2)p}$ for $k \geq 1$, i.e. $\gamma_k = \gamma_0 (\frac{n(p-1)}{(n-2)p})^k$. Since $p > \frac{n}{2}$, we have $\gamma_0 > 2$ and $\frac{n(p-1)}{(n-2)p} > 1$. We also have $\gamma_0 \frac{p}{p-1} = \frac{2n}{n-2}$. We deduce

$$\|w\|_{\gamma_k}^* \leq \left(\prod_{1 \leq i \leq k} (A_{\gamma_{i-1}} + \sqrt{2})^{\frac{2}{\gamma_i}} \right) \|w\|_{\frac{2n}{n-2}}^*. \quad (4.18)$$

Since $\frac{n(p-1)}{(n-2)p} > 1$, the product $\prod_{1 \leq i < \infty} (A_{\gamma_{i-1}} + \sqrt{2})^{\frac{2}{\gamma_i}}$ converges. We denote its value by A . Letting $k \rightarrow \infty$ we infer, on account of (4.10)

$$\|w\|_\infty \leq A \|w\|_{\frac{2n}{n-2}}^* \leq AC_1 (\|f^-\|_p^* + \|\Phi\|_{2p}^*) + A(C_1 + \sqrt{2}) \|w\|_2^*. \quad (4.19)$$

Letting $L \rightarrow \infty$ we then arrive at (4.5). ■

Remark 3 An important point in the above proof is to break the scaling invariance. Basically, the construction of the function w is not scaling invariant. More precisely, if g is transformed to $\bar{g} = \alpha g$ for a positive constant α , then (4.3) is transformed to

$$\Delta_{\bar{g}} u \geq \bar{f} + \operatorname{div} \bar{\Phi}, \quad (4.20)$$

where $\bar{f} = \alpha^{-1} f$ and $\bar{\Phi} = \alpha^{-1} \Phi$. We have

$$\|\bar{f}\|_{p,\bar{g}}^* + \|\bar{\Phi}\|_{2p,\bar{g}}^* = \alpha^{-1} (\|f\|_p^* + \|\Phi\|_{2p}^*). \quad (4.21)$$

It follows that a and hence b are not scaling invariant. The fact that the estimate (4.5) is not scaling invariant is a result of this. This unconventional feature is needed for our purpose of controlling the constants in the estimates in terms of $C_{I,N}^*(M, g)$ alone.

Remark 4 Since the estimate (4.5) is not scaling invariant, one may wonder what happens to it if one lets the above scaling factor α go to 0 or ∞ . The answer to this question is simple: the estimate deteriorates in the process. Indeed, as $\alpha \rightarrow 0$, the factor AC_1 in the estimate converges to a positive constant depending only on the dimension, but (the transformed) $\|f\|_p^* + \|\Phi\|_{2p}^*$ approaches ∞ . As $\alpha \rightarrow \infty$, AC_1 approaches ∞ and $\|f\|_p^* + \|\Phi\|_{2p}^*$ approaches 0, but the former has more weight than the latter. On the other hand, the non-invariance of the estimate allows us to vary α to obtain the optimal estimate. We do not pursue this in this paper because it is not needed for our main purpose.

The same question can be asked about the estimate in Theorem A. The answer is obviously the same.

Theorem 4.3 *Let $n \geq 3$. Assume that $u \in W^{1,2}(M)$, $f \in L^p(M)$ and $\Phi \in L^{2p}(TM)$ for some $p > \frac{n}{2}$, which satisfy*

$$\Delta u = f + \operatorname{div} \Phi \quad (4.22)$$

in the weak sense, i.e.

$$\int_M \nabla u \cdot \nabla \phi = - \int_M f \phi + \int_M \Phi \cdot \nabla \phi \quad (4.23)$$

for all $\phi \in W^{1,2}(M)$. Then we have

$$\sup_M |u - u_M| \leq C_2 (\|f\|_p^* + \|\Phi\|_{2p}^*), \quad (4.24)$$

where $C_2 = AC_1 [1 + 2 \max\{C_1, 1\}(C_1 + \sqrt{2})]$ with A and C_1 being from Lemma 4.2.

Proof. Choosing $\phi = (u - u_M)(\text{Vol}M)^{-1}$ in (4.23) we deduce by applying Lemma 4.1, as in (4.8) and (4.9)

$$\begin{aligned}
\frac{1}{\text{Vol}(M)} \int_M |\nabla u|^2 &= -\frac{1}{\text{Vol}(M)} \int_M f(u - u_M) + \frac{1}{\text{Vol}(M)} \int_M \Phi \cdot (u - u_M) \\
&\leq \|f\|_2^* \cdot \|u - u_M\|_2^* + \|\Phi\|_2^* \cdot \|\nabla u\|_2^* \\
&\leq \|f\|_p^* \cdot \|u - u_M\|_2 + \|\Phi\|_{2p}^* \cdot \|\nabla u\|_2 \\
&\leq \|f\|_p^* \cdot \|u - u_M\|_2^* + \frac{1}{2} \|\Phi\|_{2p}^{*2} + \frac{1}{2} \|\nabla u\|_2^{*2}.
\end{aligned} \tag{4.25}$$

Hence

$$\|\nabla u\|_2^{*2} \leq 2\|f\|_p^* \cdot \|u - u_M\|_2^* + \|\Phi\|_{2p}^{*2}. \tag{4.26}$$

Combining this with the Poincaré inequality (2.7) we then obtain

$$\begin{aligned}
\|u - u_M\|_2^* &\leq \frac{2(n-1)}{n-2} C_{N,I}^*(M, g) \left(\sqrt{2} \|f\|_p^{*\frac{1}{2}} \cdot \|u - u_M\|_2^{*\frac{1}{2}} + \|\Phi\|_{2p}^* \right) \\
&\leq \frac{1}{2} \|u - u_M\|_2^* + C_1^2 \|f\|_p^* + C_1 \|\Phi\|_{2p}^*,
\end{aligned} \tag{4.27}$$

where $C_1 = \frac{2(n-1)}{n-2} C_{N,I}^*(M, g)$ as before. It follows that

$$\|u - u_M\|_2^* \leq 2C_1^2 \|f\|_p^* + 2C_1 \|\Phi\|_{2p}^* \leq 2 \max\{C_1^2, C_1\} (\|f\|_p^* + \|\Phi\|_{2p}^*). \tag{4.28}$$

Combining (4.5) with $\lambda = u_M$ and (4.28) we then arrive at

$$\sup_M (u - u_M) \leq C_2 (\|f\|_p^* + \|\Phi\|_{2p}^*). \tag{4.29}$$

Replacing u by $-u$ we obtain

$$\inf_M (u - u_M) \geq -C_2 (\|f\|_p^* + \|\Phi\|_{2p}^*). \tag{4.30}$$

The estimate (4.24) follows. ■

Proof of Theorem A

Replacing f by f^- we can assume $f \leq 0$. There is a unique weak solution $v \in W^{1,2p}(M)$ of the equation

$$\Delta v = f - f_M + \text{div } \Phi \tag{4.31}$$

with $v_M = 0$. Indeed, we can minimize the functional

$$F(v) = \int_M (|\nabla v|^2 - (f - f_M)v - \Phi \cdot \nabla v) \tag{4.32}$$

for $v \in W^{1,2}(M)$ under the constraint $v_M = 0$. By the Hölder inequality and the Poincaré inequality (2.3) we have

$$F(v) \geq c\|v\|_{1,2}^2 - C(\|f - f_M\|_p^2 + \|\Phi\|_{2p}^2) \quad (4.33)$$

for some positive constants c and C , where $\|v\|_{1,2}$ denotes the $W^{1,2}$ norm of v . Hence a minimizer v exists, which is a desired solution of (4.31). Its uniqueness follows from Theorem 4.3. The property $v \in W^{1,2p}(M)$ follows from the regularity theory for elliptic operators in divergence form. By Theorem 4.3 we have

$$\sup_M v \leq C_2(\|f - f_M\|_p^* + \|\Phi\|_{2p}^*) \leq C_2(2\|f\|_p^* + \|\Phi\|_{2p}^*). \quad (4.34)$$

We set $w = u - v$. Then we have $w \in W^{1,q}(M)$ with $q = \min\{2p, \alpha\} > n$. There holds

$$\Delta w = \Delta u - \Delta v \geq f_M \quad (4.35)$$

in the weak sense, i.e.

$$\int_M \nabla w \cdot \nabla \phi \leq -f_M \int_M \phi \quad (4.36)$$

for all nonnegative $\phi \in W^{1, \frac{q}{q-1}}(M)$. Now we apply (3.6) to w to deduce

$$w(x) = w_M + \int_M \nabla_y G_0(x, y) \cdot \nabla_y w(y) dy \quad (4.37)$$

for a.e. $x \in M$. Set $\sigma = \inf_{x \neq y} G_0(x, y)$. By (3.24) there holds

$$\sigma \geq -C_0(n)C_{I,N}^*(M, g)^2 \text{vol}_g(M)^{-1}. \quad (4.38)$$

Next we set

$$G(x, y) = G_0(x, y) - \sigma. \quad (4.39)$$

We have $G(x, y) \geq 0$ and

$$w(x) = w_M + \int_M \nabla_y G(x, y) \cdot \nabla_y w(y) dy \quad (4.40)$$

for a.e. $x \in M$. By Lemma 3.1 and the fact $\frac{q}{q-1} < \frac{n}{n-1}$, (4.36) holds true with $\phi = G(x, \cdot)$ for each given $x \in M$, i.e.

$$\int_M \nabla_y G(x, y) \cdot \nabla_y w dy \leq -f_M \int_M G(x, y) dy. \quad (4.41)$$

We then deduce

$$\begin{aligned}
w(x) - w_M &\leq -f_M \int_M G(x, y) dy \\
&= |f_M| \left(\int_M G_0(x, y) dy - \sigma \text{vol}_g(M) \right) \\
&= -\sigma \text{vol}_g(M) |f_M| \\
&\leq C_0(n) C_{I,N}^*(M, g)^2 |f_M| \leq C_0(n) C_{I,N}^*(M, g)^2 \|f\|_1^* \\
&\leq C_0(n) C_{I,N}^*(M, g)^2 \|f\|_p^*
\end{aligned} \tag{4.42}$$

for a.e. $x \in M$. Since $w_M = u_M$, combining (4.34) and (4.42) yields

$$u(x) \leq u_M + (C_0(n) C_{I,N}^*(M, g) + 2C_2) \|f\|_p^* + C_2 \|\Phi\|_{2p}^* \tag{4.43}$$

for a.e. $x \in M$. We arrive at (1.6) (note that $\sup_M u$ means the essential supremum). ■

Proof of Theorem B

The estimate (1.8) follows straightforwardly from Theorem A and Theorem 2.10. ■

Proof of Theorem C

We have $\hat{\kappa}_{\hat{Ric}} = 0$. By Bonnet-Myers Theorem we have $\text{diam}_g(M) \leq \pi$. Hence Theorem B implies

$$u(x) \leq u_M + C(n, p, 0, \pi) (\|f^-\|_p^* + \|\Phi\|_{2p}^*).$$
■

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