Differential Geometry
ALM 22, pp. 311–322

Volume Comparison and its Generalizations

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Abstract

Comparison theorems are fundamental tools. In particular, the classical Bishop-Gromov volume comparison has many geometric and topological applications. Therefore it is natural to study its extensions. We will survey some applications of the volume comparison and discuss its generalizations to integral Ricci curvature, Bakry-Emery Ricci curvature, Ricci flow and, briefly, non-smooth metric measure spaces. We will also discuss some related questions.

2000 Mathematics Subject Classification: 53C20, 53C21, 53C65

Keywords and Phrases: Volume Comparison, Ricci Curvature

I met Prof. Chern several times. The first time it was in the fall of 1991 at MSRI when I was a postdoctoral member there. Long heard of Prof. Chern's kindness and being easily approachable, it was still amazing to see it first hand. Prof. Chern hosted a brunch for all the Chinese postdocs at MSRI. It is still a wonderful memory. In the summer of 1998 when Xianzhe and I visited Nankai Institute (now the Chern Institute) Prof. Weiping Zhang took us to meet with Prof. Chern at his residence there. We spent some exquisite time talking with Prof. Chern and had a terrific lunch together. It was a mesmerizing experience.

1 Volume comparison and its applications

The classical Bishop-Gromov relative volume comparison is the following monotonicity fact.

Theorem 1.1 (Bishop-Gromov's Relative Volume Comparison)**.** *Suppose M^n is a complete Riemannian manifold with* $\mathrm{Ric}_M \geq (n-1)H$*. Denote* $\mathrm{Vol}_H(B(r))$ *the*

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volume of r-ball in the model space M^n_H, the n-dimensional simply connected space form with sectional curvature H. Then for $p \in M$,

$$\frac{\mathrm{Vol}\,(\partial B(p,r))}{\mathrm{Vol}_H(\partial B(r))} \text{ and } \frac{\mathrm{Vol}\,(B(p,r))}{\mathrm{Vol}_H(B(r))} \text{ are nonincreasing in } r. \tag{1.1}$$

In particular,

$$\mathrm{Vol}\,(B(p,R)) \le \mathrm{Vol}_H(B(R)) \quad \text{for all } \ R > 0, \tag{1.2}$$

$$\frac{\mathrm{Vol}\,(B(p,r))}{\mathrm{Vol}\,(B(p,R))} \ge \frac{\mathrm{Vol}_H(B(r))}{\mathrm{Vol}_H(B(R))} \quad \text{for all } \ 0 < r \le R. \tag{1.3}$$

Moreover equality holds if and only if $B(p,R)$ is isometric to $B_H(R)$.

This is very useful because it is a global comparison. The volume of any ball is bounded above by the volume of the corresponding ball in the model, validating the intuitive picture: the bigger the curvature, the smaller the size. Moreover, and this is much less intuitive, if the volume of a big ball has a lower bound, then all smaller balls also have lower bounds. It enjoys many geometric and topological applications.

By applying volume comparison on the covering spaces and with good choice of generators one gets following results on the fundamental group and first Betti number.

Theorem 1.2 (Milnor [27]). *If M^n is complete Riemannian manifold with $\mathrm{Ric}_M \ge 0$, then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $\le n$.*

Theorem 1.3 (Gromov [18], Gallot [15]). *For closed Riemannian manifolds M^n with*

$$\mathrm{Ric}_M \ge (n-1)H, \quad \mathrm{diam}_M \le D, \tag{1.4}$$

the first Betti number $b_1(M)$ is uniformly bounded by $C(n, HD^2)$. Moreover if HD^2 is small, $b_1(M) \le n$.

In fact, very recently, Kapovitch-Wilking [20] showed that for this class of Riemannian manifolds the number of generators for the fundamental group can be uniformly bounded. Earlier the author showed this assuming in addition the conjugate radius is bounded from below [43].

Theorem 1.4 (M. Anderson [2]). *For the class of manifolds M with $\mathrm{Ric}_M \ge (n-1)H$, $\mathrm{Vol}_M \ge V$ and $\mathrm{diam}_M \le D$ there are only finitely many isomorphism types of $\pi_1(M)$.*

Remark Gromov [17] showed that a finitely generated group Γ has polynomial growth iff Γ is almost nilpotent, i.e. it contains a nilpotent subgroup of finite index. Combining this with Theorem 1.2 we know that any finitely generated subgroup of $\pi_1(M)$ of manifolds with nonnegative Ricci curvature is almost nilpotent. One naturally wonders if all finitely generated almost nilpotent groups occur as the fundamental group of some complete non-compact manifold with nonnegative

Ricci curvature. By constructing examples Wei [41] and Wilking [47] showed this is indeed true.

Remark For a *compact* Riemannian manifold with nonnegative Ricci curvature Cheeger-Gromoll [11] showed that the fundamental group is almost abelian using their splitting theorem. In the compact case, using Theorem 1.4 and volume comparison, Wei [42] generalized Milnor's result to non-collapsed almost nonnegatively Ricci curved spaces. Very recently, Kapovitch-Wilking [20] showed that the fundamental groups of almost nonnegatively Ricci curved manifolds are almost nilpotent. Its proof is much more involved, and uses strongly the work of Cheeger-Colding on the structures of Gromov-Hausdorff limits of manifolds with lower Ricci curvature bound. In fact Kapovitch-Wilking proved a Margulis Lemma for lower Ricci curvature bound, generalized Theorem 1.4 to collapsed case.

Remark When M^n has nonnegative Ricci curvature and Euclidean volume growth (i.e. $\mathrm{Vol}B(p,r) \geq cr^n$ for some $c > 0$), using a heat kernel estimate P. Li showed that $\pi_1(M)$ is finite [21]. M. Anderson also derived this using volume comparison [1].

The following geometric results can be proved directly using relative volume comparison, see e.g. [54].

Theorem 1.5 (Cheng's maximal diameter sphere theorem [12])**.** *If M^n has $\mathrm{Ric}_M \geq n-1$ and $\mathrm{diam}_M = \pi$, then M^n is isometric to the unit sphere S^n.*

Theorem 1.6 (Calabi, Yau [53])**.** *The volume of a ball in a complete non-compact manifold with nonnegative Ricci curvature grows at least linearly.*

Volume comparison has many other geometric applications, e.g. in the Gromov-Hausdorff convergence theory, in rigidity and pinching results.

Remark There is no volume comparison for Ricci curvature upper bound. In fact, Lohkamp [23] showed that for $n \geq 3$, any M^n admits a complete metric with $\mathrm{Ric}_M < 0$. Therefore Ricci curvature upper bound has no topological implications when $n \geq 3$.

2 Volume estimate for integral Ricci curvature

Why do we study integral curvature? Many geometric problems lead to integral curvatures, for example, the isospectral problems, geometric variational problems and extremal metrics, and Chern-Weil's formula for characteristic numbers. Since integral curvature bound is much weaker than pointwise curvature bound, one naturally asks what geometric and topological results can be extended to integral curvature.

The volume comparison theorem can be generalized to an integral Ricci lower bound [33], see also [16, 51]. For convenience we introduce some notation.

For each $x \in M^n$ let $\lambda(x)$ denote the smallest eigenvalue for the Ricci tensor $\mathrm{Ric} : T_xM \to T_xM$, and $\mathrm{Ric}_-^H(x) = \max\{0, (n-1)H - \lambda(x)\}$ — the amount of

Ricci curvature lies below $(n-1)H$. Let

$$\|\mathrm{Ric}_-^H\|_p(R) = \sup_{x\in M}\left(\int_{B(x,R)} (\mathrm{Ric}_-^H)^p \, dvol\right)^{\frac{1}{p}}. \tag{2.1}$$

$\|\mathrm{Ric}_-^H\|_p$ measures the amount of Ricci curvature lying below $(n-1)H$ in the L^p sense. Clearly $\|\mathrm{Ric}_-^H\|_p(R) = 0$ iff $\mathrm{Ric}_M \ge (n-1)H$.

We have

Theorem 2.1 (Relative Volume Estimate, Petersen-Wei [33]). *Let $x \in M^n, H \in \mathbb{R}$ and $p > \frac{n}{2}$ be given, then there is a constant $C(n,p,H,R)$ which is nondecreasing in R such that if $r \le R$ and when $H > 0$ assume that $R \le \frac{\pi}{2\sqrt{H}}$ we have*

$$\left(\frac{\mathrm{Vol}\, B(x,R)}{\mathrm{Vol}_H(B(R))}\right)^{\frac{1}{2p}} - \left(\frac{\mathrm{Vol}\, B(x,r)}{\mathrm{Vol}_H(B(r))}\right)^{\frac{1}{2p}} \le C(n,p,H,R)\cdot\left(\|\mathrm{Ric}_-^H\|_p(R)\right)^{\frac{1}{2}}. \tag{2.2}$$

Furthermore when $r = 0$ we obtain

$$\mathrm{Vol}\, B(x,R) \le \left(1 + C(n,p,H,R)\cdot\left(\|\mathrm{Ric}_-^H\|_p(R)\right)^{\frac{1}{2}}\right)^{2p} \mathrm{Vol}_H(B(R)). \tag{2.3}$$

Note that when $\|\mathrm{Ric}_-^H\|_p(R) = 0$, this gives the Bishop-Gromov relative volume comparison, Theorem 1.1. Let $\overline{\|\mathrm{Ric}_-^H\|_p}(R)$ be the normalized L^p-norm,

$$\overline{\|\mathrm{Ric}_-^H\|_p}(R) = \sup_{x\in M}\left(\frac{1}{\mathrm{Vol}B(x,R)}\int_{B(x,R)} (\mathrm{Ric}_-^H)^p \, dvol\right)^{\frac{1}{p}}.$$

From (2.2) one can estimate the volume doubling constant when the normalized L^p-norm of the Ricci curvature below $(n-1)H$ is small.

Corollary 2.2 (Volume Doubling Estimate). *Given $\alpha < 1$ and $p > \frac{n}{2}$, there is an $\epsilon = \epsilon(n,p,H,\alpha,R) > 0$ such that if $\overline{\|\mathrm{Ric}_-^H\|_p}(R) < \epsilon$, then for all $x \in M^n$ and $r_1 < r_2 \le R$ (assume $R \le \frac{\pi}{2\sqrt{H}}$ when $H > 0$),*

$$\frac{\mathrm{Vol}B(x,r_1)}{\mathrm{Vol}B(x,r_2)} \ge \alpha\,\frac{\mathrm{Vol}_H(r_1)}{\mathrm{Vol}_H(r_2)}. \tag{2.4}$$

With these, for $p > \frac{n}{2}$, many pinching and compactness results for $\|\mathrm{Ric}_-^H\|_p(r) = 0$ (Ricci curvature bounded from below pointwisely) can be extended to the case when $\|\mathrm{Ric}_-^H\|_p(r)$ is very small (Ricci curvature bounded from below in L^p), showing a gap phenomenon. See e.g. [16, 33, 13, 34, 14].

On the other hand, unlike pointwise Ricci curvature lower bound, Ricci curvature bounded from below in L^p does not automatically lift to the covering spaces. Therefore certain topological implications, such as those on the fundamental group, for Ricci curvature bounded from below in L^p does not follow immediately since we need to apply volume comparison on the covering space. By using a star-shaped region in the covering space and Petersen-Wei's pointwise estimate [33] Aubry showed the mean of the integral Ricci curvature on the geodesic balls of the covering space can be controlled by the mean of the manifold, allowing several topological extensions of pointwise Ricci curvature to integral Ricci curvature, see [32, 3, 19, 4]. We conjecture that the recent result of Kapovitch-Wilking [20] can also be extended to integral Ricci curvature.

Conjecture 2.3. *For $n \in \mathbb{N}$, $p > \frac{n}{2}$, there exists a constant $\epsilon(n,p) > 0$ such that if a compact Riemannian manifold M^n has $Diam_M \leq 1$ and $\overline{\|\mathrm{Ric}_-^0\|}_p(1) \leq \epsilon(n,p)$ then the fundamental group of M is almost nilpotent.*

3 Comparison for Bakry-Emery Ricci Tensor

The Bakry-Emery Ricci tensor is a Ricci tensor for smooth metric measure spaces, which are Riemannian manifold with a measure conformal to the Riemannian measure. Formally a smooth metric measure space is a triple $(M^n, g, e^{-f} dvol_g)$, where M is a complete n-dimensional Riemannian manifold with metric g, f is a smooth real valued function on M, and $dvol_g$ is the Riemannian volume density on M. This is also sometimes called a manifold with density. These spaces occur naturally as smooth collapsed limits of manifolds with lower Ricci curvature bound under the measured Gromov-Hausdorff convergence. Recall

Definition 3.1. $(X_i, \mu_i) \overset{\mathrm{mGH}}{\longrightarrow} (X_\infty, \mu_\infty)$ *(compact) in measured Gromov-Hausdorff sense if for all sequences of continuous functions $f_i : X_i \to \mathbb{R}$ converging to $f_\infty : X_\infty \to \mathbb{R}$, we have*

$$\int_{X_i} f_i d\mu_i \to \int_{X_\infty} f_\infty d\mu_\infty.$$

Example 3.2. *Let $(M^n \times F^N, g_\epsilon)$ be a product manifold with warped product metric $g_\epsilon = g_M + (\epsilon e^{-f})^2 g_F$, where f is a function on M. Then up to a constant scaling in the measure $(M \times F, g_\epsilon, \widetilde{dvol_{g_\epsilon}})$ converges to $(M^n, g_M, e^{-Nf} dvol_{g_M})$ under the measured Gromov-Hausdorff convergence. Here $\widetilde{dvol_{g_\epsilon}} = dvol_{g_\epsilon}/\mathrm{Vol}(B(\cdot, 1))$ is a renormalized Riemannian measure.*

Therefore we have, as $\epsilon \to 0$,

$$(M^n \times F^N, \widetilde{dvol_{g_\epsilon}}) \overset{\mathrm{mGH}}{\longrightarrow} (M^n, e^{-f} dvol_{g_M}),$$

where $g_\epsilon = g_M + (\epsilon e^{-\frac{f}{N}})^2 g_F$. By O'Neill's formula, the Ricci curvature of the warped product metric g_ϵ in the M direction is

$$\mathrm{Ric}_M + \mathrm{Hess} f - \frac{1}{N} df \otimes df.$$

Hence for smooth metric measure spaces $(M^n, g, e^{-f}dvol_g)$, the corresponding Ricci tensor is the N-Bakry-Emery Ricci tensor

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{N} df \otimes df \quad \text{for } 0 < N \leq \infty. \tag{3.1}$$

When $N = \infty$ we denote $\mathrm{Ric}_f = \mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess} f$. Note that when f is a constant function $\mathrm{Ric}_f^N = \mathrm{Ric}$ for all N and we can take $N = 0$ in this case. Moreover, if $N_1 \geq N_2$ then $\mathrm{Ric}_f^{N_1} \geq \mathrm{Ric}_f^{N_2}$ so that $\mathrm{Ric}_f^N \geq \lambda g$ implies $\mathrm{Ric}_f \geq \lambda g$.

The Bakry-Emery Ricci curvature bounded from below (for N finite and infinite) has a natural extension to non-smooth metric measure spaces [25, 39, 40] and diffusion operators [5], see Section 5. Moreover, the equation $\mathrm{Ric}_f = \lambda g$ for some constant λ is exactly the gradient Ricci soliton equation, which plays an important role in the theory of Ricci flow; the equation $\mathrm{Ric}_f^N = \lambda g$, for N positive integer, corresponds to the warped product metric on $M \times_{e^{-\frac{f}{N}}} F^N$ is Einstein, where F^N is an N-dimensional Einstein manifold with suitable Einstein constant, see [9].

One naturally asks if the Bishop-Gromov volume comparison extends to the Bakry-Emery Ricci tensor. When N is finite, the answer is yes, one only needs to compare to the "model space" of dimension $n + N$. Denote $\mathrm{Vol}_f(B(p, R)) = \int_{B(p,R)} e^{-f} dvol$.

Theorem 3.3 (Volume comparison for N-Bakry-Emery, Qian [35])**.** *If* $\mathrm{Ric}_f^N \geq (n + N - 1)H$, *then* $\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_H^{n+N}(R)}$ *is nonincreasing in* R.

When N is infinite, this is not true as the following example shows.

Example 3.4. *Let* $M = \mathbb{R}^n$ *be equipped with Euclidean metric* g_0, $f(x_1, \cdots, x_n) = x_1$. *Then* $\mathrm{Ric}_f = \mathrm{Ric} = 0$. *But* $\mathrm{Vol}_f(B(0, r)) = \int_{B(0,r)} e^{-f} dvol$ *is of exponential growth.*

On the other hand when f or ∇f is bounded we still have nice volume comparisons.

Theorem 3.5 (Volume Comparison for ∞-Bakry-Emery, Wei-Wylie [46])**.** *Let* $(M^n, g, e^{-f}dvol_g)$ *be a complete smooth metric measure space with* $\mathrm{Ric}_f \geq (n-1)H$. *Fix* $p \in M^n$.
a) If $\partial_r f \geq -a$ *along all minimal geodesic segments from* p *then for* $R \geq r > 0$ *(assume* $R \leq \pi/2\sqrt{H}$ *if* $H > 0$*)* ,

$$\frac{\mathrm{Vol}_f(B(p, R))}{\mathrm{Vol}_f(B(p, r))} \leq e^{aR} \frac{\mathrm{Vol}_H^n(R)}{\mathrm{Vol}_H^n(r)}. \tag{3.2}$$

Moreover, equality holds if and only if the radial sectional curvatures are equal to H *and* $\partial_r f \equiv -a$. *In particular if* $\partial_r f \geq 0$ *and* $\mathrm{Ric}_f \geq 0$ *then* M *has* f*-volume growth of degree at most* n.
b) If $|f(x)| \leq k$ *then for* $R \geq r > 0$ *(assume* $R \leq \pi/4\sqrt{H}$ *if* $H > 0$*),*

$$\frac{\mathrm{Vol}_f(B(p, R))}{\mathrm{Vol}_f(B(p, r))} \leq \frac{\mathrm{Vol}_H^{n+4k}(R)}{\mathrm{Vol}_H^{n+4k}(r)}. \tag{3.3}$$

In particular, if f is bounded and $\mathrm{Ric}_f \geq 0$ *then M has polynomial f-volume growth.*

Remark In fact, with a more careful estimate, one can show that if f is bounded and $\mathrm{Ric}_f \geq 0$ then M has polynomial f-volume growth of degree at most n, see [52]. Therefore the exact same statement of Milnor's result (Theorem 1.2) holds for $\mathrm{Ric}_f \geq 0$.

Remark The relative volume comparison Theorem 3.5 also implies theorems of Gromov [18] (Theorem 1.3) and Anderson [2] (Theorem 1.4) extend to the case when Ric_f is bounded below and f or $|\nabla f|$ is uniformly bounded. Similarly Theorem 1.6 extends to noncompact manifolds with $\mathrm{Ric}_f \geq 0$ when f is bounded. See [46] for detail.

Recently Bakry-Emery Ricci tensor and its applications in Ricci solitons have been actively investigated and there are many interesting results. Other than the ones mentioned above, see [24, 6, 30, 7, 22, 49, 9, 28, 8, 26, 48] among others, see also [45] for a short survey in the direction of comparison theorem.

When f is bounded, $\mathrm{Ric}_f \geq 0$ gives the same topological obstruction on the fundamental group as a metric with $\mathrm{Ric} \geq 0$. This raises the following question.

Question 3.6. *If M^n is a compact Riemannian manifold with a measure such that* $\mathrm{Ric}_f \geq 0$*, does M^n have another metric on it with* $\mathrm{Ric} \geq 0$*?*

For noncompact smooth metric measure space, one can ask if Milnor's result is true without any assumption on f. Namely

Question 3.7. *If M^n is a Riemannian manifold with a measure such that* $\mathrm{Ric}_f \geq 0$*, does any finitely generated subgroup of its fundamental group have polynomial growth of degree $\leq n$?*

4 Volume Comparison in Ricci Flow

Perelman's reduced volume monotonicity [30], a fundamental and powerful tool in his work on Thurston's geometrization conjecture, is a generalization to Ricci flow of Bishop-Gromov's volume comparison. In fact Perelman gave a heuristic argument that volume comparison on an infinite dimensional space (incorporating the Ricci flow) gives the reduced volume monotonicity. We present his argument [30, §6.1] here.

Recall $(M, g(\tau))$ is a backward Ricci flow when

$$\frac{\partial g}{\partial \tau} = 2Ric_{g_\tau}.$$

For a curve $\gamma(\tau), 0 \leq \tau_1 \leq \tau \leq \tau_2$ on M, Perelman considers the energy function

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}\left[S(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2_{g(\tau)}\right] d\tau,$$

where $S(\gamma(\tau))$ is the scalar curvature of the metric $g(\tau)$ at $\gamma(\tau)$. Fix $x \in M$, for $y \in M, \tau \in (0, T)$, the $\mathcal{L}$-distance is

$$L(y, \tau) = \inf\{\mathcal{L}(\gamma) | \gamma : [0, \tau] \to M, \gamma(0) = x, \gamma(\tau) = y\}$$

and the reduced distance is $l(y,\tau) = \frac{1}{2\sqrt{\tau}} L(y,\tau)$. Now the reduced volume is given by

$$\tilde{V}_x(\tau) = \int_{(M,g(\tau))} \tau^{-n/2} e^{-l(y,\tau)} dy.$$

Theorem 4.1 (Perelman, [30]). *$\tilde{V}_x(\tau)$ is monotonically decreasing in τ.*

Heuristically this can be derived by applying Bishop-Gromov volume comparison to infinite dimensional space. Let $\tilde{M} = \mathbb{R}^+ \times M \times \mathbb{S}^N$ be equipped with the metric:

$$\tilde{g} = \left(\frac{N}{2\tau} + S\right) d\tau^2 + g(\tau) + 2N\tau g_{\alpha\beta}, \tag{4.1}$$

where $g(\tau)$ is a backward Ricci flow on M, $g_{\alpha\beta}$ is the standard metric on $\mathbb{S}^N$.

Then mod N^{-1}, the distance in $\tilde{M}$ corresponds to $\mathcal{L}$-distance in M and $\tilde{M}$ is Ricci flat. We derive the first statement here, for second statement, see [44] for detail.

The shortest geodesic between $(0,x)$ and (τ,q) is orthogonal to $\mathbb{S}^N$. Hence the length of such curve $\gamma(\tau)$ is

$$\begin{aligned} l(\gamma(\tau)) &= \int_0^{\tau(q)} \sqrt{\frac{N}{2\tau} + S + |\dot{\gamma}_M(\tau)|^2}\, d\tau \\ &= \int_0^{\tau(q)} \sqrt{\frac{N}{2\tau}} \sqrt{1 + \frac{2\tau(S + |\dot{\gamma}_M(\tau)|^2)}{N}}\, d\tau \\ &= \sqrt{2N\tau(q)} + \frac{1}{\sqrt{2N}} \int_0^{\tau(q)} \sqrt{\tau}(S + |\dot{\gamma}_M(\tau)|^2) d\tau + O(N^{-\frac{3}{2}}). \end{aligned} \tag{4.2}$$

When N is big the shortest geodesic would minimize $\int_0^{\tau(q)} \sqrt{\tau}(S+|\dot{\gamma}_M(\tau)|^2)d\tau$, corresponds to the $\mathcal{L}$-distance.

Since $(\tilde{M},\tilde{g})$ is Ricci flat mod N^{-1}, we can compare its area of geodesics sphere to the model space Euclidean space.

Consider the metric sphere $\partial B((0,x), \sqrt{2N\tau(y)}) \subset \tilde{M}$. If $(\tau,q) \in \partial B((0,x), \sqrt{2N\tau(y)})$, i.e. $d((0,x),(\tau,q)) = \sqrt{2N\tau(y)}$, by (4.2)

$$\sqrt{2N\tau(y)} = \sqrt{2N\tau(q)} + \frac{1}{\sqrt{2N}} L(q,\tau) + O(N^{-\frac{3}{2}}).$$

Hence the geodesics sphere $\partial B((0,x), \sqrt{2N\tau(y)})$ is $O(N^{-1})$ close to the hypersurface $\tau = \tau(y)$ and its area is given by

$$\begin{aligned} \mathrm{Vol}(\partial B((0,x), \sqrt{2N\tau(y)})) &= \int_{(M,g(\tau))} (2N\tau(q))^{N/2} \mathrm{Vol}(\mathbb{S}^N) dvol_q \\ &= \mathrm{Vol}(\mathbb{S}^N)(2N)^N \int_{(M,g(\tau))} \left(\sqrt{\tau(y)} - \frac{1}{2N} L(q,\tau) + O(N^{-2})\right)^N dvol_q. \end{aligned}$$

Compare this to the geodesics sphere in the model space $\mathbb{R}^{n+N+1}$, we have the ratio

$$\begin{aligned}
&\frac{\mathrm{Vol}(\partial B((0,x),\sqrt{2N\tau(y)}))}{\mathrm{Vol}(\partial B((N+n+1,0,\sqrt{2N\tau(y)}))} \\
&\quad = \frac{(2N\sqrt{\tau})^N \mathrm{Vol}(\mathbb{S}^N)\int_{(M,g(\tau))}\left(1-\frac{L}{2\sqrt{\tau}}\frac{1}{N}+O(N^{-2})\right)^N dvol}{(\sqrt{2N\tau})^{N+n}\mathrm{Vol}(\mathbb{S}^{N+n})} \\
&\quad = C(n,N)\int_{(M,g(\tau))}\tau^{-\frac{n}{2}}\left(1-\frac{L}{2\sqrt{\tau}}\frac{1}{N}\right)^N dvol + O(N^{-1}).
\end{aligned}$$

As $N \to \infty$, $\left(1-\frac{L}{2\sqrt{\tau}}\frac{1}{N}\right)^N \to e^{-\frac{L}{2\sqrt{\tau}}}$. By (1.1) the reduced volume $\tilde{V}(\tau) = \int_{(M,g(\tau))}\tau^{-n/2}e^{-l}dvol$ is monotonically decreasing in τ.

Reduced volume monotonicity has rigidity result [30], it will be very interesting to study its stability case.

5 Volume compariosn for non-smooth metric measure spaces

In [39, 40, 25] Sturm and Lott-Villani independently defined notion of Ricci curvature lower bounds, called curvature dimension condition, for a metric space with a measure. The curvature dimension conditions correspond to the convexity of certain entropy functions on the space of probability measures. Any Gromov-Hausdorff limits of manifolds with Ricci curvature bounded from below satisfy the curvature dimension condition. For smooth metric measure spaces the curvature dimension conditions exactly correspond to the Bakry-Emery Ricci tensors in Section 3 are bounded from below, which give Bishop-Gromov volume comparison when the dimension is finite.

For a Riemannian manifold Ricci curvature lower bound is not characterized by Bishop-Gromov volume comparison. In fact, the local Bishop-Gromov volume comparison (only true for small r) is characterized by scalar curvature lower bound. Instead Ricci curvature lower bound is characterized by an infinitesimal Bishop-Gromov volume comparison [29], see also [40, 10]. One can use this to define Ricci curvature lower bound for metric space with measure, but there are ambiguity in dimension.

With volume comparison it seems one readily get information about fundamental group again. Again we need to apply it to the universal. Hence we need to first investigate if the universal cover of spaces satisfying the curvature dimension condition exists. If it exits, whether the universal cover is simply connected. For Gromov-Hausdorff limits of manifolds with Ricci curvature bounded from below, Sormani-Wei [37, 38] showed the universal cover exists.

Acknowledgement: Part of this note represents author's lecture at RTG seminar, Michgan State University 2007 and four lectures given at summer school on "New approaches to curvature, Les Diablerets, Switzerland, 2008. I would like to thank the organizers for the opportunity to have given these lectures. The research is also partially supported by NSF grant DMS-0806016.

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