Metrics of positive Ricci curvature on vector bundles over nilmanifolds

Igor Belegradek Guofang Wei*

Abstract

We construct metrics of positive Ricci curvature on some vector bundles over tori (or more generally, over nilmanifolds). This gives rise to the first examples of manifolds with positive Ricci curvature which are homotopy equivalent but not homeomorphic to manifolds of nonnegative sectional curvature.

1 Introduction

According to the soul theorem of J. Cheeger and D. Gromoll, a complete open manifold of nonnegative sectional curvature is diffeomorphic to the total space of the normal bundle of a compact totally geodesic submanifold. Furthermore, a finite cover of any compact manifold with $\sec \geq 0$ is diffeomorphic to the product $C \times T$ where T is a torus, and C is simply-connected with $\sec \geq 0$. (The same is actually true when sec is replaced by Ric.)

It was shown in [ÖW94, BKb, BKa] that a majority of vector bundles over $C \times T$ admit no metric with $\sec \geq 0$ provided $\dim(T)$ is sufficiently large. For example, any vector bundle over a torus whose total space admits a metric with $\sec \geq 0$ becomes trivial in a finite cover, and there are only finitely many such bundles in each rank.

It is then a natural question which of the bundles can carry metrics with $\mathrm{Ric} \geq 0$ or $\mathrm{Ric} > 0$. J. Nash and L. Berard-Bergery [Nas79, BB78] constructed metrics with $\mathrm{Ric} > 0$ on every vector bundle of rank ≥ 2 over a compact manifold C with $\mathrm{Ric}(C) > 0$. Their method generally fails if C only has $\mathrm{Ric}(C) \geq 0$. In fact, the metrics produced in [Nas79, BB78] have the property that the bundle projection is a Riemannian submersion, while according to [BKb, 2.2], any vector bundle over a torus with $\mathrm{Ric} \geq 0$ on the total space becomes trivial in a finite cover provided the bundle projection is a Riemannian submersion.

Recall that a *nilmanifold* is the quotient of a simply-connected nilpotent group by a discrete subgroup. Any nilmanifold is diffeomorphic to the product of a compact nilmanifold and a Euclidean space [Mal51]. Compact nilmanifolds can be described as iterated principal circle bundles [TO97]. Here is our main result (see also 6.1 for a more technical version).

Theorem 1.1. *Let L be a complex line bundle over a nilmanifold. Then for all sufficiently large k, the total space of the Whitney sum of k copies of L admits a complete Riemannian metric of positive Ricci curvature.*

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Note that if the total spaces $E(\xi)$, $E(\eta)$ of vector bundles of ξ, η over a smooth aspherical manifold M admit complete metrics with $\mathrm{Ric} \geq 0$, then so does the total space of $\xi \oplus \eta \oplus TM$ because $E(\xi \oplus \eta \oplus TM)$ is the covering of $E(\xi) \times E(\eta)$ corresponding to the subgroup $d_* \pi_1(M)$ where $d \colon M \to M \times M$ is the diagonal embedding. Since complete nilmanifolds are aspherical and parallelizable, Theorem 1.1 generalizes, up to taking sum with some trivial bundle, to vector bundles with tori as structure groups (the structure group of a vector bundle is a k-torus iff the bundle is a Whitney sum of k complex line bundles and a trivial bundle). Combining 6.1 with [BKb, 1.2], we deduce the following.

Corollary 1.2. *Let C be a point or a compact manifold with $Ric \geq 0$, and let T be a torus of dimension ≥ 4. Then for all sufficiently large k, there is an infinite sequence of rank k vector bundles over $C \times T$ whose total spaces are pairwise nonhomeomorphic, admit complete Riemannian metrics with $Ric > 0$, and are not homeomorphic to complete Riemannian manifolds with* $\sec \geq 0$.

Corollary 1.2 gives a new type of examples of manifolds with $\mathrm{Ric} > 0$. These manifolds are topologically similar but not homeomorphic to manifolds with $\sec \geq 0$. We refer to [Gro93] for the survey of known examples. Now the following question is natural.

Question 1.3. *Do most vector bundles over tori (or more generally compact manifolds with $Ric \geq 0$ or* $\sec \geq 0$*) admit metrics with $Ric \geq 0$, or $Ric > 0$?*

Note that by M. Anderson's first Betti number estimate [And90], no vector bundle of rank ≤ 2 over a torus admits a metric with $\mathrm{Ric} > 0$. This however seems to be the only known obstruction. Theorem 1.1 also gives metrics of $\mathrm{Ric} > 0$ on some nontrivial vector bundles over many nilmanifolds as follows.

Corollary 1.4. *Let N be a nilmanifold such that there exists a class in $H^2(N, \mathbb{Z})$ with nontorsion cup-square. Then for all sufficiently large k, there is an infinite sequence of rank k vector bundles over N whose total spaces are pairwise nonhomeomorphic and admit complete Riemannian metrics with $Ric > 0$.*

The existence of a 2-class with a nontorsion cup-square is equivalent, by A.4, to the existence of a complex line bundle L over N such that for any k, the Whitney sum of k copies of L is nontrivial as a real vector bundle. In A.3 we give an example of a compact nilmanifold of dimension 5 for which such an L does not exist.

Metrics of positive Ricci curvature on vector bundles over nilmanifolds are interesting in their own right. Apparently, besides Anderson's growth estimate [And90], no obstructions are known to the existence of such metrics.

Here is the idea of the proof of Theorem 1.1. The second author showed in [Wei88] that if G is a simply-connected nilpotent Lie group, then $G \times \mathbb{R}^k$ has a warped product metric with $\mathrm{Ric} > 0$ such that G lies in the isometry group. In this paper we start with a similar metric on $G \times \mathbb{C}^k$ where G has a central one-parametric subgroup Q. Let Q act on G by left multiplication, fix an epimorphism $Q \to S^1$, and a diagonal isometric action of S^1 on $\mathbb{C}^k$. We then show that the Riemannian submersion metric on $(G \times \mathbb{C}^k)/Q$ has $\mathrm{Ric} > 0$ for all large enough k. (Note that, generally, Ricci curvature may decrease under

Riemannian submersions, namely, there is an error term in the O'Neill submersion formula for Ricci curvature. The error term always vanishes for submersions with minimal fibers, however, in our case the fibers clearly have different volume on the $\mathbb{C}^k$-factor so they are not minimal. We prove that $(G \times \mathbb{C}^k)/Q$ has $\mathrm{Ric} > 0$ by being able to control the error term when k is sufficiently large.) The obvious G-action on $(G \times \mathbb{C}^k)/Q$ is isometric so the quotients of $(G \times \mathbb{C}^k)/Q$ by discrete subgroups of G also carry metrics with $\mathrm{Ric} > 0$. The bundles described in 1.1 can be obtained as $(P \times \mathbb{C}^k)/Q$ where P is a principal circle bundle associated with L. Finally, 1.1 follows from the fact that a principal circle bundle over a nilmanifold is again a nilmanifold.

The structure of the paper is as follows. In sections 2 and 3 we define the metric, and choose a coordinate system to compute the Ricci curvature. This computation is done in sections 4 and 5. Applications are discussed in section 6. In Appendix we prove some facts on the existence of line bundles over nilmanifolds.

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2 Defining the metric

Let G be a simply-connected nilpotent Lie group with the Lie algebra g. Fix an arbitrary central element $X_1 \in g$ and use the ascending central series for g to choose a basis $X_1, \cdots, X_n$ for g such that $[X, X_i] \in \mathrm{span}\{X_1, \cdots, X_{i-1}\}$ for all $X \in g$. In particular,

$$[X_1, X_i] = 0 \text{ for all } i = 1, \cdots, n. \tag{2.1}$$

Equip $G \times \mathbb{C}^k$ with the warped product metric constructed in [Wei88]. Namely, let

$$g = g_r + dr^2 + f^2(r) ds^2_{2k-1}$$

be the metric on $G \times \mathbb{C}^k$, where $f(r) = r(1 + r^2)^{-1/4}$, ds^2_{2k-1} is the canonical metric on the sphere S^{2k-1} (the one induced by the inclusion of the unit sphere into $\mathbb{C}^k$), and g_r is a special family of almost flat left invariant metrics on G defined as follows.

Let $Y_i = X_i/h_i(r)$, where

$$h_i(r) = (1 + r^2)^{-\alpha_i}, \tag{2.2}$$

and $\alpha_i = 2^{n-i+1} + 2^{-1-i} - \frac{1}{2}$. (This value of α_i is obtained by choosing $\alpha_n = 2^{-1-n} + \frac{3}{2}$ in [Wei88].) Then let g_r be the left-invariant metric on G such that $\{Y_1, \cdots, Y_n\}$ form an orthonormal basis. As we explain in remark 3.2 below, the Ricci tensor of g_r satisfies

$$|\mathrm{Ric}_G(Y_i, Y_j)| \leq c(1 + r^2)^{-1}, \tag{2.3}$$

where c is a constant depending on n and the structure constants. Clearly, g is a complete metric on $G \times \mathbb{C}^k$ whose isometry group contains G. By [Wei88], $\mathrm{Ric}(g) > 0$ for all sufficiently large k.

For a nonzero integer m, let $\varphi^m \colon \mathbb{R} \to U(1)$ be the homomorphism given by $\varphi^m(q) = e^{imq}$. Define a sequence of isometric $\mathbb{R}$-actions Φ_m on $G \times \mathbb{C}^k$ by

$$\Phi_m(q)(x, u) = (\exp(qX_1)x, \varphi^m(q)u) \text{ for any } x \in G, u \in \mathbb{C}^k. \tag{2.4}$$

Equip $(G\times\mathbb{C}^k)/\Phi_m(\mathbb{R})$ with the Riemannian submersion metric. Note that the isometric G-action on $G\times\mathbb{C}^k$ given by $g(h,z)=(gh,z)$ induces an isometric G-action on the quotient $(G\times\mathbb{C}^k)/\Phi_m(\mathbb{R})$; the kernel of the action is exactly $K_m=\exp(\frac{2\pi}{m}\mathbb{Z}X_1)$ so that G/K_m acts effectively.

3 Setting up the computation

Let Ric denote the Ricci curvature on the total space $G\times\mathbb{C}^k$ of the submersion and $\check{\text{Ric}}$ the Ricci curvature on the base $(G\times\mathbb{C}^k)/\Phi_m(\mathbb{R})$. Then for any vector fields X,Y on the base, the Ricci curvatures are related by the following (see [Bes87, p.244])

$$\check{\text{Ric}}(X,Y)=\text{Ric}(X,Y)+2\langle A_X,A_Y\rangle+\langle TX,TY\rangle-\frac{1}{2}(\langle\nabla_X S,Y\rangle+\langle\nabla_Y S,X\rangle), \tag{3.5}$$

where A,T are the O'Neill tensors, and S is the mean curvature of the fiber (for precise definition, see, e.g. [Bes87, p.239]). In our case $\langle A_X,A_Y\rangle=\langle A_XW,A_YW\rangle$, $\langle TX,TY\rangle=\langle T_WX,T_WY\rangle$ where W is defined in (3.7).

We want to show that for all sufficiently large k, the Riemannian submersion metric on $(G\times\mathbb{C}^k)/\Phi_m(\mathbb{R})$ has positive Ricci curvature. To prove this, it suffices to control the error term $\check{\text{Ric}}(X,Y)-\text{Ric}(X,Y)$ in (3.5) coming from the mean curvature tensor of the fiber of the $\mathbb{R}$-action.

For the rest of the computation, fix a point (g_*,r_*,s_*) of $G\times\mathbb{C}^k$ and show that the horizontal space tangent to the point has all Ricci curvatures positive.

At the point s_* of the sphere (S^{2k-1},ds^2_{2k-1}) choose an orthonormal frame $V_1,\cdots,V_{2k-1}$ such that V_1 is tangent to the orbit of the action Φ_m restricted to S^{2k-1}, and $V_2,\cdots,V_{2k-1}$ are horizontal lifts of an orthonormal basis $\hat{V}_2,\cdots,\hat{V}_{2k-1}$ on CP^{k-1} which diagonalizes the Ricci tersor of CP^{k-1} at the projection of s_* to CP^{k-1}. (Here we view S^{2k-1} as the Hopf fibration S^1 over CP^{k-1}). Since the fibers of the Hopf fibration are totally geodesic, one has $\nabla_{V_1}V_1=0$ with respect to ds^2_{2k-1}, so

$$\langle V_1,[V_1,V_j]\rangle=0 \text{ for all } j=1,\cdots,2k-1. \tag{3.6}$$

Let $U_j=V_j/f(r)$. Then $\partial_r=\frac{\partial}{\partial r},U_1,\cdots U_{2k-1},Y_1,\cdots,Y_n$ form an orthonormal basis of g on $G\times\mathbb{C}^k$ in a neighborhood of (g_*,r_*,s_*). The vector field X_1+mV_1 is tangent to the orbits of the $\mathbb{R}$-action Φ_m on $G\times\mathbb{C}^k$. Let

$$W=(X_1+mV_1)/({h_1}^2+m^2f^2)^{1/2}=(h_1Y_1+mfU_1)/({h_1}^2+m^2f^2)^{1/2} \tag{3.7}$$

be its unit vector. Then $\partial_r,U_2,\cdots U_{2k-1},Y_2,\cdots,Y_n,W'$ form an orthonormal basis of $(G\times\mathbb{C}^k)/\Phi_m(\mathbb{R})$, where

$$W'=(mfY_1-h_1U_1)/({h_1}^2+m^2f^2)^{1/2}. \tag{3.8}$$

The mean curvature vector S of the fiber is $S=(\nabla_WW)^H$, the horizontal projection of ∇_WW. Using Koszul's formula for the Riemannian connection (the one used to prove its existence) and (2.1), (3.6), we have

$$\nabla_{Y_1}Y_1=-\frac{h_1'}{h_1}\partial_r,\quad \nabla_{U_1}U_1=-\frac{f'}{f}\partial_r,\quad \nabla_{Y_1}U_1=\nabla_{U_1}Y_1=0.$$

Since any function of r has zero derivatives in the directions Y_1, U_1, Koszul's formula implies

$$S = -\frac{h_1 h_1' + m^2 f f'}{{h_1}^2 + m^2 f^2} \partial_r. \tag{3.9}$$

The Ricci curvatures of the total space $G \times \mathbb{C}^k$ are given by [Wei88, Wei89] and remark 3.2 below.

$$\begin{aligned} \mathrm{Ric}(\partial_r, U_j) = 0 & \quad (1 \le j \le 2k-1), \\ \mathrm{Ric}(Y_i, \partial_r) = \mathrm{Ric}(Y_i, U_j) = 0 & \quad (1 \le i \le n,\ 1 \le 2k-1), \\ |\mathrm{Ric}(Y_i, Y_j)| \le c(1+r^2)^{-1}, & \quad (i \neq j, 1 \le i, j \le n), \end{aligned}$$

where c is a constant depending only on n and the structure constants.

$$\begin{aligned} \mathrm{Ric}(\partial_r, \partial_r) &= -\sum_{i=1}^n \frac{h_i''(r)}{h_i(r)} - (2k-1)\frac{f''(r)}{f(r)}, \\ &= \left\{ -\sum_{i=1}^n 2\alpha_i[(2\alpha_i+1)r^2 - 1] + (2k-1)\frac{r^2+6}{4} \right\} / (1+r^2)^2. \quad (3.10) \\ \mathrm{Ric}(Y_i, Y_i) &= -\frac{h_i''(r)}{h_i(r)} - (2k-1)\frac{h_i'(r) f'(r)}{h_i(r) f(r)} + \mathrm{Ric}_G(Y_i) - \sum_{i \neq j} \frac{h_i'(r) h_j'(r)}{h_i(r) h_j(r)} \\ &\ge \{-2\alpha_i[(2\alpha_i+1)r^2 - 1] + (2k-1)\alpha_i(2+r^2) \\ &\quad -c(1+r^2) - \sum_{i\neq j} 4\alpha_i\alpha_j r^2\}/(1+r^2)^2 \quad (1 \le i \le n), \quad (3.11) \\ \mathrm{Ric}(U_j, U_j) &= -\frac{f''(r)}{f(r)} + (2k-2)\frac{1 - f'(r)^2}{f(r)^2} - \sum_{i=1}^n \frac{h_i'(r) f'(r)}{h_i(r) f(r)} \\ &\qquad (1 \le j \le 2k-1). \quad (3.12) \end{aligned}$$

The Ricci tensor is positive definite since each of the diagonal terms of the Ricci tensor has a positive term with a factor k which controls the negative terms [Wei89] and dominates the off-diagonal terms when $k \ge k_0(n, \alpha_i, c)$.

Remark 3.1. It was claimed in [Wei88, Wei89] that $\mathrm{Ric}(Y_i, Y_j) = 0$ if $i \neq j$. Actually, this is only true when the basis $X_1, \ldots, X_n$ of the Lie algebra of G satisfies

$$\langle [X_i, X_j], [X_j, X_k] \rangle = 0 \ \text{ for all } \ i \neq k, \tag{3.13}$$

which holds in many cases but it is unclear whether such a basis can be always found. Say, for the nilpotent Lie algebra of strictly upper triangular matrices, (3.13) is satisfied by the so-called triangular basis given by strictly upper triangular matrices with 1 at one spot and 0's elsewhere (see [Wei89]). Note that by Ado's theorem any nilpotent Lie algebra is isomorphic to a subalgebra of the algebra of strictly upper triangular matrices, but it is not always possible to choose a basis of the subalgebra to be a subset of the triangular basis. Other examples of nilpotent Lie algebras with a basis satisfying (3.13) include all the two-step nilpotent Heisenberg algebras (see e.g. [CG90]), all nilpotent Lie algebra in dim ≤ 5 [Dix58], and 22 out of 24 nilpotent Lie algebra in dimension 6 [Mor58].

Remark 3.2. However, all the results of [Wei88] are true as stated because $|\mathrm{Ric}(Y_i, Y_j)| \le c(1+r^2)^{-1}$ which suffices for the positivity of Ricci curvature. The above estimate is obtained by computing $\mathrm{Ric}(Y_i, Y_j)$ in the basis $\partial_r, U_1, \cdots U_{2k-1}, Y_1, \cdots, Y_n$. By [Wei89] the terms $\langle R(Y_i, \partial_r)\partial_r, Y_j\rangle$, $\langle R(Y_i, U_k)U_k, Y_j\rangle$ vanish, and

$$\begin{aligned} \langle R(Y_i, Y_k)Y_k, Y_j\rangle &= \frac{3}{4} h_i^{-1}(r) h_k^{-2}(r) h_j^{-1}(r) \langle [X_i, X_k], [X_k, X_j]\rangle, \quad \text{if } i < k < j, \\ &= \frac{1}{2} h_i^{-1}(r) h_k^{-2}(r) h_j^{-1}(r) \langle [X_i, X_k], [X_k, X_j]\rangle \quad \text{if } i < j < k, \text{ or } k < i < j. \end{aligned}$$

Clearly $h_i^{-1}(r) h_k^{-2}(r) h_j^{-1}(r) |\langle [X_i, X_k], [X_k, X_j]\rangle| \le c(1+r^2)^{-1}$, for some c depending only on n and the structure constants. Thus, the same estimate holds for $|\mathrm{Ric}(Y_i, Y_j)|$.

4 Controlling the error term

In this section we show that the extra terms $\check{\mathrm{Ric}}(X,Y) - \mathrm{Ric}(X,Y)$ in (3.5) (to which we refer to as the *error*) are controlled by some positive term from the Ricci curvature of the total space.

To compute the error for $\check{\mathrm{Ric}}(\partial_r, \partial_r)$, note that

$$\begin{aligned} \langle T\partial_r, T\partial_r\rangle &= \langle T_W \partial_r, T_W \partial_r\rangle \\ &= \langle (\nabla_W \partial_r)^V, (\nabla_W \partial_r)^V\rangle \\ &= \langle \nabla_W \partial_r, W\rangle^2 = \langle W, [\partial_r, W]\rangle^2. \end{aligned}$$

Since $[\partial_r, X_1 + mV_1]$=0, it follows from (3.7) that

$$[\partial_r, W] = -\frac{h_1 h_1' + m^2 f f'}{{h_1}^2 + m^2 f^2} W. \tag{4.14}$$

So

$$\langle T\partial_r, T\partial_r\rangle = \left(\frac{h_1 h_1' + m^2 f f'}{{h_1}^2 + m^2 f^2}\right)^2.$$

Also

$$\begin{aligned} \langle A_{\partial_r}, A_{\partial_r}\rangle &= \langle A_{\partial_r} W, A_{\partial_r} W\rangle = \langle (\nabla_{\partial_r} W)^H, (\nabla_{\partial_r} W)^H\rangle \\ &= \langle \nabla_{\partial_r} W, W'\rangle^2 = \frac{1}{4}\langle W, [\partial_r, W']\rangle^2, \end{aligned}$$

where for the third equality we use Koszul's formula for the Riemannian connection to show that $(\nabla_{\partial_r} W)^H$ only has W'-component, and for the last equality we combine Koszul's formula with (4.14). Now from (3.8), we compute

$$[\partial_r, W'] = -\frac{h_1 {h_1}' + m^2 f f'}{h_1^2 + m^2 f^2} W' + \frac{f' h_1 - f h_1'}{({h_1}^2 + m^2 f^2)^{1/2}} \cdot \left(\frac{m}{{h_1}^2} X_1 + \frac{1}{f^2} V_1\right). \tag{4.15}$$

Thus

$$\langle A_{\partial_r}, A_{\partial_r}\rangle = \left(m \frac{f h_1' - h_1 f'}{{h_1}^2 + m^2 f^2}\right)^2.$$

On the other hand, since by Koszul's formula $\nabla_{\partial_r}\partial_r = 0$, we get

$$\begin{aligned}\langle \nabla_{\partial_r} S, \partial_r\rangle &= -\left(\frac{h_1 h_1' + m^2 f f'}{{h_1}^2 + m^2 f^2}\right)' \\ &= -\frac{({h_1'}^2 + h_1 h_1'' + m^2 f'^2 + m^2 f f'')({h_1}^2 + m^2 f^2) - 2(h_1 {h_1}' + m^2 f f')^2}{({h_1}^2 + m^2 f^2)^2}.\end{aligned}$$

Hence, the error for $\check{\mathrm{Ric}}(\partial_r, \partial_r)$ is

$$\begin{aligned}&2\langle A_{\partial_r}, A_{\partial_r}\rangle + \langle T\partial_r, T\partial_r\rangle - \langle \nabla_{\partial_r} S, \partial_r\rangle \\ &= \frac{3m^2(h_1' f - f' h_1)^2}{({h_1}^2 + m^2 f^2)^2} + \frac{h_1 h_1'' + m^2 f f''}{{h_1}^2 + m^2 f^2} \geq \frac{h_1 h_1'' + m^2 f f''}{{h_1}^2 + m^2 f^2} \\ &= \frac{\frac{h_1''}{h_1} + m^2 \frac{f''}{f}(\frac{f}{h_1})^2}{1 + m^2(\frac{f}{h_1})^2} = \frac{2\alpha_1\left[(2\alpha_1+1)r^2 - 1\right] - m^2(\frac{3}{2} + \frac{1}{4}r^2)r^2(1+r^2)^{2^{n+1}-1}}{\left[1 + m^2 r^2(1+r^2)^{2^{n+1}-1}\right](1+r^2)^2}.\end{aligned}$$

Here in the last equality we used some of the the following equations.

$$\left(\frac{f}{h_1}\right)^2 = r^2(1+r^2)^{2^{n+1}-1}, \tag{4.16}$$

$$\frac{h_i'}{h_i} = -\frac{2\alpha_i r}{1+r^2}, \qquad \frac{h_i''}{h_i} = \frac{2\alpha_i[(2\alpha_i+1)r^2 - 1]}{(1+r^2)^2}, \tag{4.17}$$

$$\frac{f'}{f} = \frac{1 + \frac{r^2}{2}}{r(1+r^2)}, \qquad \frac{f''}{f} = -\frac{\frac{3}{2} + \frac{1}{4}r^2}{(1+r^2)^2}. \tag{4.18}$$

By (3.10), $\mathrm{Ric}(\partial_r, \partial_r)$ has a positive term $(2k-1)\frac{r^2+6}{4}/(1+r^2)^2$. Clearly,

$$(2k-1)\frac{r^2+6}{4} > -\frac{2\alpha_1\left[(2\alpha_1+1)r^2 - 1\right] - m^2(\frac{3}{2} + \frac{1}{4}r^2)r^2(1+r^2)^{2^{n+1}-1}}{1 + m^2 r^2(1+r^2)^{2^{n+1}-1}}$$

for $k \geq \frac{2}{3}\alpha_1 + \frac{3}{2}$. Note that $\alpha_1 = 2^n - \frac{1}{4}$. Therefore, $\check{\mathrm{Ric}}(\partial_r, \partial_r)$ is positive for all integers $k \geq k_0 + \frac{2^{n+1}}{3} + \frac{4}{3}$, where k_0 has been chosen before Remark 3.1.

For $\check{\mathrm{Ric}}(Y_i, Y_i)$, $i = 2, \cdots, n$, the error $\geq -\langle \nabla_{Y_i} S, Y_i\rangle$. We can compute

$$\begin{aligned}\langle \nabla_{Y_i} S, Y_i\rangle &= -\langle Y_i, [S, Y_i]\rangle \\ &= -\frac{h_i'}{h_i}\cdot\frac{h_1 h_1' + m^2 f f'}{h_1^2 + m^2 f^2} = -\frac{h_i'}{h_i}\cdot\frac{\frac{h_1'}{h_1} + m^2\frac{f'}{f}(\frac{f}{h_1})^2}{1 + m^2(\frac{f}{h_1})^2} \\ &= \frac{2\alpha_i r^2\left[-2\alpha_1 + m^2(1 + \frac{r^2}{2})(1+r^2)^{2^{n+1}-1}\right]}{1 + m^2 r^2(1+r^2)^{2^{n+1}-1}}/(1+r^2)^2,\end{aligned}$$

which again is controlled by the positive term $(2k-1)\alpha_i(2+r^2)/(1+r^2)^2$ in $\mathrm{Ric}(Y_i, Y_i)$ (see (3.11)) when $k \geq 1$.

For $\check{\mathrm{Ric}}(U_j,U_j)$, $j=2,\cdots,2k-1$, the error $\geq -\langle \nabla_{U_j} S, U_j\rangle$. Again

$$\begin{aligned}\langle \nabla_{U_j} S, U_j\rangle &= -\langle U_j, [S,U_j]\rangle \\ &= -\frac{f'}{f}\cdot\frac{h_1h_1'+m^2ff'}{{h_1}^2+m^2f^2} \\ &= -\frac{(1+\frac{r^2}{2})(-2\alpha_1+m^2(1+\frac{r^2}{2})(1+r^2)^{2^{n+1}-1})}{(1+m^2r^2(1+r^2)^{2^{n+1}-1})(1+r^2)^2}.\end{aligned}$$

This is controlled by the positive term $(2k-2)\frac{1-f'^2}{f^2}$ in $\mathrm{Ric}(U_j,U_j)$ (see (3.12)) when $k\geq \frac{2^{n+1}}{3}+1$. Note that

$$\frac{1-f'^2}{f^2}\geq \frac{\frac{3}{2}+r^2}{(1+r^2)^2}.$$

For $\check{\mathrm{Ric}}(W',W')$, the error $\geq -\langle \nabla_{W'}S,W'\rangle = \langle W',[S,W']\rangle$. Denote $g=\frac{h_1h_1'+m^2ff'}{{h_1}^2+m^2f^2}$. Then $S=-g\partial_r$, and $[S,W']=-g[\partial_r,W']$. Applying (4.15) we have

$$\begin{aligned}\langle \nabla_{W'}S,W'\rangle &= -g^2+g\cdot\frac{f'h_1-fh_1'}{{h_1}^2+m^2f^2}\left[\frac{m^2f}{h_1}-\frac{h_1}{f}\right] \\ &= -g\frac{\frac{f'}{f}+m^2\frac{h_1'}{h_1}(\frac{f}{h_1})^2}{1+m^2(\frac{f}{h_1})^2} \\ &= \frac{-\left(\frac{h_1'}{h_1}+m^2\frac{f'}{f}(\frac{f}{h_1})^2\right)\left(\frac{f'}{f}+m^2\frac{h_1'}{h_1}(\frac{f}{h_1})^2\right)}{(1+m^2(\frac{f}{h_1})^2)^2}.\end{aligned}$$

Now

$$\begin{aligned}&-(1+r^2)^2\left(\frac{h_1'}{h_1}+m^2\frac{f'}{f}\left(\frac{f}{h_1}\right)^2\right)\left(\frac{f'}{f}+m^2\frac{h_1'}{h_1}\left(\frac{f}{h_1}\right)^2\right) \\ &\quad= -\left(-2\alpha_1+m^2(1+\frac{r^2}{2})(1+r^2)^{2^{n+1}-1}\right)\left(1+\frac{r^2}{2}-2\alpha_1m^2r^4(1+r^2)^{2^{n+1}-1}\right) \\ &\quad= \left[2\alpha_1m^4r^4(1+\frac{r^2}{2})(1+r^2)^{2^{n+2}-2}-m^2\left(4\alpha_1^2r^4+(1+\frac{r^2}{2})^2\right)(1+r^2)^{2^{n+1}-1}+2\alpha_1(1+\frac{r^2}{2})\right]\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{Ric}(W',W') &= \mathrm{Ric}\left(\frac{mfY_1-h_1U_1}{({h_1}^2+m^2f^2)^{1/2}},\frac{mfY_1-h_1U_1}{({h_1}^2+m^2f^2)^{1/2}}\right) \\ &= \frac{1}{{h_1}^2+m^2f^2}\left[m^2f^2\mathrm{Ric}(Y_1,Y_1)+{h_1}^2\mathrm{Ric}(U_1,U_1)\right] \\ &= \frac{1}{1+m^2(\frac{f}{h_1})^2}\left[m^2\left(\frac{f}{h_1}\right)^2\mathrm{Ric}(Y_1,Y_1)+\mathrm{Ric}(U_1,U_1)\right].\end{aligned}$$

Now the positive term $\frac{1}{1+m^2(\frac{f}{h_1})^2}(2k-1)\alpha_1m^2(\frac{f}{h_1})^2(2+r^2)/(1+r^2)^2$ in $\mathrm{Ric}(W',W')$ controls $\frac{1}{(1+m^2(\frac{f}{h_1})^2)^2}\left[2\alpha_1m^4r^4(1+\frac{r^2}{2})(1+r^2)^{2^{n+2}-2}\right]/(1+r^2)^2$ when $k\geq 1$, and the positive term

$\frac{1}{1+m^2(\frac{f}{h_1})^2}(2k-2)\frac{1-f'^2}{f^2}$ in $\mathrm{Ric}(W',W')$ controls $\frac{1}{(1+m^2(\frac{f}{h_1})^2)^2}2\alpha_1(1+\frac{r^2}{2})/(1+r^2)^2$ when $k \geq \frac{2^{n+1}}{3}+1$.

5 Computing mixed terms

Here we show that the errors from the mixed terms either vanish, or controlled by the positive diagonal term of the Ricci curvature of the total space.

First, from (3.9) one can compute that $[S,X]$ only has component in the X direction for $X=\partial_r$, Y_i or U_j. Also $[S,W']$ is a linear combination of Y_1 and U_1. So the last term

$$-\frac{1}{2}\left(\langle \nabla_X S, Y\rangle + \langle \nabla_Y S, X\rangle\right)$$

in (3.5) of the error of all mixed terms is zero.

Second, note that $T_W X$ vanishes for $X = Y_i$, U_i, or W'. (Indeed, $T_W X = (\nabla_W X)^V$ so $\langle T_W X, W\rangle = \langle \nabla_W X, W\rangle = \langle [W,X], W\rangle$, and $[W,X]=0$ since any function of r has zero derivative in the direction of X.) Thus, the T-component of the error always vanishes for the mixed terms, and it remains to control the A-component.

By the remark preceding (4.15), we have $A_{\partial_r} W = (\nabla_{\partial_r} W)^H = \langle \nabla_{\partial_r} W, W'\rangle W'$. By Koszul's formula $\langle A_{Y_i} W, W'\rangle = 0$ since $[W,X]=0$ for $X=Y_i$, or W'. So the error is zero and $\check{\mathrm{Ric}}(\partial_r, Y_i) = 0$ for $i = 2,\cdots,n$. Similarly, $\check{\mathrm{Ric}}(\partial_r, U_j) = 0$, for $j=2,\cdots,2k-1$.

Now the error for $\check{\mathrm{Ric}}(Y_i,Y_j)$ with $i\neq j$, $i,j = 2,\cdots,n$ is equal to $2\langle A_{Y_i}, A_{Y_j}\rangle$. By Koszul's formula $\langle A_{Y_i} W, Y_k\rangle = \langle W, [Y_k, Y_i]\rangle/2$ and $A_{Y_i}W$ has zero components in U_j, ∂_r, W'. Hence, $\langle A_{Y_i}, A_{Y_j}\rangle$ is the sum of the terms $\langle W,[Y_k,Y_i]\rangle\langle W,[Y_k,Y_j]\rangle/4$. Since W has length one, the absolute value of each of the terms is bounded above by the product $|[Y_i,Y_k]|\cdot|[Y_k,Y_j]|$, which is bounded above by $c(1+r^2)^{-1}$ as in 3.2. Hence, the error is controlled by the positive diagonal terms of the total space.

Similarly, by Koszul's formula $\langle A_{U_i} W, U_k\rangle = \langle W, [U_k, U_i]\rangle/2$ and $A_{U_i}W$ has zero components in Y_j, ∂_r, W'. Hence, $\langle A_{U_i}, A_{U_j}\rangle = \sum_k \langle W,[U_k,U_i]\rangle\langle W,[U_k,U_j]\rangle/4$, which is proportional to $\sum_k \langle V_1,[V_k,V_i]\rangle\langle V_1,[V_k,V_j]\rangle/2 = \mathrm{Ric}_{CP^{k-1}}(\hat{V}_i,\hat{V}_j) - \mathrm{Ric}_{S^{2k-1}}(V_i,V_j) = 0$ if $i\neq j$ and $i,j=2,\cdots,2k-1$ by our choice of the basis in section 3. Thus, $\check{\mathrm{Ric}}(U_i,U_j)=0$ if $i\neq j$ and $i,j=2,\cdots,2k-1$.

By above A_{Y_i}, A_{U_j} are orthogonal since they only have nonzero components in Y_k's and U_k's, respectively. Thus, $\check{\mathrm{Ric}}(Y_i,U_j)$, $\;i=2,\cdots,n,\; j=2,\cdots,2k-1$ is zero.

Finally, the errors for

$$\begin{aligned}&\check{\mathrm{Ric}}(\partial_r, W'),\\ &\check{\mathrm{Ric}}(Y_i, W'), \quad i=2,\cdots,n,\\ &\check{\mathrm{Ric}}(U_j, W'), \quad j=2,\cdots,2k-1\end{aligned}$$

vanish since $\nabla_{W'}W$ only has a nonzero component in ∂_r, and $\langle \nabla_{Y_i} W, \partial_r\rangle = \langle \nabla_{U_j} W, \partial_r\rangle = 0$.

This completes the proof that the manifold $(G\times\mathbb{C}^k)/\Phi_m(\mathbb{R})$ has positive Ricci curvature for all integers $k\geq k_0+\frac{2^{n+1}}{3}+\frac{4}{3}$.

Remark 5.1. Note that our k depends only on n, c, α_i, and is not optimal. There are other choices of α_i (or even $h_i(r)$ which work in special cases) that can make k smaller (cf. [Wei89]).

6 Metrics on vector bundles over nilmanifolds

Here is a technical version of 1.1 which we need for the applications.

Theorem 6.1. *Let N be a nilmanifold and $\alpha \in H^2(N, \mathbb{Z})$. For an integer m, let L_m be a (unique up to isomorphism) complex line bundle with the first Chern class $c_1(L_m) = m\alpha$. Let $\xi_{k,m}$ be the Whitney sum of k copies of L_m. Then there exists $K_0 = K_0(N, \alpha)$ such that for each $k \geq K_0$ the total space of $\xi_{k,m}$ admits a complete Riemannian metric with $Ric > 0$.*

Proof. By [Mal51] N is diffeomorphic to the product $N_c \times \mathbb{R}^s$ where N_c is a compact nilmanifold. Let P be a principal $U(1)$-bundle over N_c with the first Chern class equal to α. (Recall that c_1 gives a bijection between the set $[X, BU(1)]$ of $U(1)$-bundles over a finite cell complex X and the group $H^2(X, \mathbb{Z})$.)

Being a principal circle bundle over a compact nilmanifold, P is a compact nilmanifold so by [Mal51] $P = G_c/\Gamma$ where G_c is a simply-connected nilpotent Lie group, and Γ is a discrete subgroup of G. Let z be the (central) element of Γ corresponding to a fiber of P. By [CG90, 5.1.5], z lies in the center of G_c. Let Q be the (necessarily central) one-parametric subgroup containing z. Consider the nilpotent Lie group $G = G_c \times \mathbb{R}^s$ and think of Q, Γ as subgroups of G. The subgroup $Q \cap \Gamma$ of Γ is generated by z (else z would be a power of some element of Γ which is then projected to a torsion element of the torsion free group $\pi_1(N) = \Gamma/\langle z \rangle$). Thus, the group $\Gamma/\langle z \rangle$ is a discrete subgroup of G/Q with factor space N. Now G/Γ gets the structure of a principal $Q/\langle z \rangle \cong U(1)$-bundle over N which is isomorphic to $P \times \mathbb{R}^s$. (Since the bundle projections induce the same maps of fundamental groups, the bundles are fiber homotopy equivalent which implies isomorphism for principal circle bundles.)

Now let X_1 be a nonzero element of the Lie algebra of G tangent to Q. Use X_1 to define the $\mathbb{R}$-action Φ_m as in (2.4). Thus, there exists a positive $K_0 = K_0(P)$ such that for all $k \geq K_0$, the manifold $X = (G \times \mathbb{C}^k)/\Phi_m(\mathbb{R})$ has a complete G-invariant metric with $\text{Ric} > 0$, so we can consider the quotient of X by the isometric action of $\Gamma \leq G$. Note that the kernel of the G-action is K_m, hence the kernel of the Γ-action is $K_m \cap \Gamma = \langle z \rangle$.

The actions of G and $\Phi_m(\mathbb{R})$ on $G \times \mathbb{C}^k$ commute, so X/Γ is the quotient of $G/\Gamma \times \mathbb{C}^k$ by the drop of Φ_m. The drop of Φ_m can be described as the $Q/\langle z \rangle$-action on the G/Γ-factor, and the same diagonal $U(1)$-action on $\mathbb{C}^k$.

Thus, X/Γ is the total space of a complex vector bundle over N. The bundle is the Whitney sum of k copies of the same line bundle which we denote L'_m. Finally, we show that the first Chern class of L'_m is $m\alpha$. Indeed, since L'_1 is constructed via the standard $U(1)$-action on $\mathbb{C}$, we get $c_1(L'_1) = \alpha$. Look at the associated circle bundles $S(L'_1)$ and $S(L'_m)$. There is a covering $S(L'_1) \to S(L'_m)$ which is an m-fold covering on every fiber. Then the classifying map for $S(L'_m)$ can be obtained as the classifying map for $S(L'_1)$ followed by the selfmap of BS^1 induced by the m-fold selfcover of S^1. By the homotopy sequence of the universal S^1-bundle, this selfmap of BS^1 induces the multiplication by m on the second homotopy group $\pi_2(BS^1)$. By Hurewitz it induces the same map on $H_2(BS^1,\mathbb{Z})$ and hence on $H^2(BS^1,\mathbb{Z})$. Since the first Chern class of the universal bundle generates $H^2(BS^1,\mathbb{Z})$, we deduce $c_1(L'_m)=mc_1(L'_1)=m\alpha$.

Since the first Chern class determines (the isomorphism type of) the complex line bundle, L'_m is isomorphic to L_m, and the proof is complete. □

Proof of 1.4. Let $\alpha\in H^2(N,\mathbb{Z})$ be a class with a nontorsion cup-square. Let L_m be a (unique up to isomorphism) complex line bundle over N with $c_1(L)=m\alpha$. By 6.1 for all sufficiently large k the Whitney sum $\xi_{k,m}$ of k copies of L_m has a complete metric with $\mathrm{Ric}>0$. The total Pontrjagin class of $\xi_{k,m}$ can be computed as

$$p(\xi_{k,m})=c(\xi_{k,m}\oplus\bar{\xi}_{k,m})=c(L_m)^k c(\bar{L}_m)^k=(1+c_1(L_m))^k(1-c_1(L_m))^k=(1-c_1(L_m)^2)^k,$$

hence the first Pontrjagin class $\xi_{k,m}$ is given by $p_1(\xi_{k,m})=-kc_1(L_m)^2=-km^2\alpha^2$.

It remains to show that the total spaces $E(\xi_{k,m})$ of $\xi_{k,m}$ are pairwise nonhomeomorphic for different values of $|m|$. Being the quotient of a Lie group by a discrete subgroup, N is parallelizable (because the tangent bundle to any Lie group has a left-invariant trivialization which descends to quotients by discrete subgroups). Hence, the tangent bundle $TE(\xi_{k,m})$ has first Pontrjagin class $-km^2\alpha^2$. Assume that $h\colon E(\xi_{k,m'})\to E(\xi_{k,m})$ is a homeomorphism. Since rational Pontrjagin classes are topologically invariant, $-km^2h^*(\alpha^2)$ is equal to $-k(m')^2\alpha^2$ up to elements of finite order. Thus, the projections of the classes to $H^4(N,\mathbb{Z})/Tors$ are mapped to each other by the automorphism h^* of $H^4(N,\mathbb{Z})/Tors$. Since any automorphism of a finitely generated, free abelian group preserves multiplicities of primitive elements, $m=\pm m'$, and we are done. □

Proof of 1.2. Since $\dim(T)\geq 4$, we can find $\alpha\in H^2(T,\mathbb{Z})$ with a nontorsion cup-square (see A.1). Furthermore, we can choose α so that $\alpha^3=0$, say, by taking α to be the pullback of a class with a nontorsion cup-square via an obvious projection of T onto the 4-torus.

As in proof of 1.4, we find a vector bundle $\xi_{k,m}$ over T such that $p_1(\xi_{k,m})=-km^2\alpha^2$ and $\mathrm{Ric}(E(\xi_{k,m}))>0$. Since $\alpha^3=0$, all the higher Pontrjagin classes of $\xi_{k,m}$ vanish. Since $p_1(\xi_{k,m})$ is a nontorsion class, $\xi_{k,m}$ does not become stably trivial after passing to a finite cover because finite covers induce injective maps on rational cohomology.

Now the product $C\times E(\xi_{k,m})$ has a (product) metric with $\mathrm{Ric}\geq 0$ which by [BB86] can be assumed of $\mathrm{Ric}>0$ after replacing $E(\xi_{k,m})$ with $E(\xi_{k,m})\times\mathbb{R}^3$ (or equivalently, after replacing $\xi_{k,m}$ by its Whitney sum with a trivial rank 3 bundle). By [BKb, 1.2] the manifold $C\times E(\xi_{k,m})$ admits no metric with $\sec\geq 0$.

It remains to show that the manifolds $C\times E(\xi_{k,m})$ are pairwise nonhomeomorphic for different values of $|m|$. The tangent bundle to $C\times E(\xi_{k,m})$ is isomorphic to $TC\times TE(\xi_{k,m})$. Let $i\geq 0$ be the largest number such that the ith integral Pontrjagin class $p_i(TC)$ is nontorsion. Since T is parallelizable, the bundle $TE(\xi_{k,m})$ has first integral Pontrjagin class $-km^2\alpha^2$, and zero higher Pontrjagin classes. Up to elements of order two, the total Pontrjagin classes satisfy the product formula $p(TC \times TE(\xi_{k,m})) = p(TC) \times p(TE(\xi_{k,m}))$. In particular, $p_{i+1}(TC \times TE(\xi_{k,m})) = p_i(TC) \times p_1(TE(\xi_{k,m}))$ is a nontorsion class since it is a product of nontorsion classes.

Assume that $h\colon C \times E(\xi_{k,m'}) \to C \times E(\xi_{k,m})$ is a homeomorphism. Since rational Pontrjagin classes are topologically invariant, h^* maps $p_i(TC) \times (-km^2\alpha^2)$ to $p_i(TC) \times (-k(m')^2\alpha^2)$ up to elements of finite order. Thus, the projections of the classes to $H^{4(i+1)}(C \times T, \mathbb{Z})/Tors$ are mapped to each other by the automorphism h^* of $H^{4(i+1)}(N, \mathbb{Z})/Tors$. Since any automorphism of a finitely generated, free abelian group preserves multiplicities of primitive elements, $m = \pm m'$, and we are done. □

Remark 6.2. The above proof applies verbatim when C is a complete manifold with $\mathrm{Ric} \geq 0$ which is the total space of a vector bundle over a compact smooth manifold. Say, we can take C to be any Ricci positively curved vector bundles of [Nas79, BB78].

A Existence of line bundles over nilmanifolds

Let N be a nilmanifold. In this appendix we discuss when there is $\alpha \in H^2(N, \mathbb{Z})$ with nontorsion cup-square α^2. As we note in A.4, this is equivalent to the existence of a complex line bundle L such that, for any k, the Whitney sum of k copies of L is nontrivial as a real vector bundle. Since any nilmanifold is diffeomorphic to the product of a compact nilmanifold and a Euclidean space, it suffices to discuss the case when N is compact which we assume from now on. Clearly, one necessary condition for the existence of α as above is $\dim(N) \geq 4$, however it is not sufficient, in general, as we show in A.3.

Example A.1. One situation in which $\dim(N) \geq 4$ is sufficient is when N is a torus, or more generally, a symplectic nilmanifold. (Indeed, since nondegeneracy is an open condition, the symplectic form ω can be approximated by a nondegenerate rational form ω' (where "rational" means that the cohomology class of ω' lies in the image of $H^2(N, \mathbb{Q}) \to H^2(N, \mathbb{R})$). Finally, we take α to be the cohomology class of a suitable integer multiple of ω'.) We refer to [TO97] for concrete examples of symplectic nilmanifolds. Of course, if N carries a cohomology 2-class with a nontorsion cup-square, then so does any product $N \times N'$.

Example A.2. If N_1, N_2 are arbitrary compact nilmanifolds of dimension ≥ 2, then there is $\alpha \in H^2(N_1 \times N_2, \mathbb{Z})$ with nontorsion cup-square. To see that first note if N is a compact nilmanifold of dimension ≥ 2, then the second Betti number of N is positive. (Indeed, the conclusion is obvious if N is a 2-torus, so we can assume $\dim(N) \geq 3$. Since N can be obtained as an iterated principal circle bundle, there is a π_1-surjective map of N onto a 2-torus. In particular, the first Betti number of N is ≥ 2. Since $\dim(N) \geq 3$, the second Betti number satisfies $b_2(N) \geq b_1(N)^2/4$ [FHT97] which implies $b_2(N) > 0$.) Then given compact nilmanifolds N_1, N_2 of dimensions ≥ 2, we can choose a class $\omega_i \in H^2(N_i, \mathbb{Z})$ so that there is $F_i \in H_2(N_i, \mathbb{Z})$ with $\langle \omega_i, F_i \rangle \neq 0$. Let $p_i\colon N_1 \times N_2 \to N_i$ be the projection. Now one of the classes $p_1^*\omega_1$, $p_2^*\omega_2$, $p_1^*\omega_1 + p_2^*\omega_2$ has a nontorsion cup-square because if $p_1^*\omega_1^2$, $p_2^*\omega_2^2$ are torsion classes, then up to elements of finite order $(p_1^*\omega_1 + p_2^*\omega_2)^2 = \omega_1 \times \omega_2$, and $\langle \omega_1 \times \omega_2, F_1 \times F_2 \rangle = \langle \omega_1, F_1 \rangle \cdot \langle \omega_2, F_2 \rangle \neq 0$.

Counterexample A.3. We now present a compact 5-dimensional nilmanifold N such that any class in $H^2(N,\mathbb{Z})$ has a zero cup-square. Let x_1, x_2, x_3, x_4 be generators of $H^1(T,\mathbb{Z})$ where T is the 4-torus. Let $p\colon\ N \to T$ be a principal circle bundle over T with Euler class $e = x_1x_2 + x_3x_4$. Look at the Gysin sequence of the bundle with integer coefficients:

$$H^0(T,\mathbb{Z}) \overset{\cup e}{\to} H^2(T,\mathbb{Z}) \overset{p^*}{\to} H^2(N,\mathbb{Z}) \to H^1(T,\mathbb{Z}) \overset{\cup e}{\to} H^3(T,\mathbb{Z}) \to \dots$$

By a direct computation $H^1(T,\mathbb{Z}) \overset{\cup e}{\to} H^3(T,\mathbb{Z})$ is an isomorphism (this also follows from the Hard Lefschetz theorem since e is proportional to the standard Kähler form for T). By exactness p^* is onto. Now take an arbitrary $y \in H^2(N,\mathbb{Z})$ and find x with $y = p^*(x)$. Since $H^4(T,\mathbb{Z}) \cong \mathbb{Z}$, the class x^2 is a rational multiple of e^2, hence $y^2 = (p^*x)^2 = p^*(x^2)$ is a rational multiple of $p^*(e^2) = (p^*e)^2 = 0$ where the last inequality holds because e is in the kernel of p^*.

Lemma A.4. *Given a space X homotopy equivalent to a finite cell complex, the following are equivalent:*

(i) there exists $\alpha \in H^2(X,\mathbb{Z})$ whose cup-square α^2 is nontorsion;

(ii) there exists a complex line bundle L over X such that, for any k, the Whitney sum of k copies of L is nontrivial as a real vector bundle.

Proof. Assume (i), and find a complex line bundle L over X with first Chern class $c_1(L) = \alpha$. Let ξ_k be the Whitney sum of k copies of L. Hence $p_1(\xi_k) = -kc_1(L)^2 = -k\alpha^2$ is nontorsion. This proves (ii).

Now assume (ii) and take $\alpha = c_1(L)$. Arguing by contradiction, assume α^2 is a torsion class, that is $k\alpha^2 = 0$ for some $k > 0$. As above we get $p_1(\xi_k) = -k\alpha^2 = 0$, hence all rational Pontrjagin classes vanish. In particular, the Pontrjagin character ph of ξ_k vanishes. Recall that ph defines an isomorphism of $\widetilde{KO}(X) \otimes \mathbb{Q}$ and $\oplus_{i>0} H^{4i}(X,\mathbb{Q})$, thus ξ_k has finite order in $\widetilde{KO}(X)$. In particular, for some $m > 0$, the Whitney sum of m copies of ξ_k is trivial, as a real bundle. The contradiction proves (i). □

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DEPARTMENT OF MATHEMATICS, 253-37, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125, USA

email: `ibeleg@its.caltech.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SANTA BARBARA, SANTA BARBARA, CA 93106, USA

email: `wei@math.ucsb.edu`