Introductory Lecture

April 3, 2018

1 Introduction

In 1859 Riemann studied the zeros of the Riemann zeta function $\zeta(s)$ and linked them to the distribution of prime numbers. In 1927 Pólya introduced a family of functions $H_t(z)$, parametrized by $t \in \mathbb{R}$, with H_0 giving the Riemann zeta function. The function H_t can be viewed as the evolution of the function H_0 under the backwards heat flow. The zeros of H_0 are linked to the zeros of ζ which are intimately related to prime numbers. This quarter we study the zeros of the entire functions $H_t(z)$; do they carry arithmetical information? If so, how?

2 The functions $H_t(z)$

In his 1859 paper B. Riemann [6] introduced his Riemann xi function ξ, defined as

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

where ζ is the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1.$$

The motivation for his xi function is that the critical zeros $\mathrm{Re}(s) = 1/2$ of ζ corresponds to the real zeros of ξ, up to a change of variables [1], and vice versa. Let $H_0 : \mathbb{C} \to \mathbb{C}$ denote the function

$$H_0(z) := \frac{1}{8}\xi\left(\frac{1}{2} + \frac{iz}{2}\right).$$

The Riemann Hypothesis is equivalent to $H_0(z)$ having only real zeros. The function $H_0(z)$ is even, entire, and has the Fourier representation

$$H_0(z) = \int_0^\infty \Phi(u)\cos(zu)du,$$

[1] Thank you to G. Sarajian for the correction.

where

$$\Phi(u) := \sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) \exp(-\pi n^2 e^{4u})$$

is a real function with very fast decay at infinity. In 1927 G. Pólya [5] introduced a family of functions $H_t : \mathbb{C} \to \mathbb{C}$, $t \in \mathbb{R}$, by the formula

$$H_t(z) := \int_0^{\infty} \exp(tu^2)\Phi(u)\cos(zu)du.$$

This function H_t can be viewed as the evolution of the function H_0 under the backwards heat equation $\partial_t H_t(z) = -\partial_{zz} H_t(z)$. In 1950 N.C. de Bruijn [2] showed that if $t \geq 1/2$, then $H_t(z)$ has only real zeros. In 1976 C.M. Newman [4] showed that there exists a constant $\Lambda \in (-\infty, 1/2]$, now known as the de Bruijn-Newman constant, such that

$$H_t(z) \text{ has only real zeros if and only if } t \geq \Lambda.$$

The Riemann Hypothesis is then equivalent to $\Lambda \leq 0$. Newman [4] remarked that

> The problem of determining whether a Fourier transform has only real zeros arises in two rather disparate areas of mathematics: number theory and mathematical physics. In number theory, the problem is intimately associated with the Riemann hypothesis, while in mathematical physics it is closely connected with the Lee-Yang theorem of statistical mechanics and quantum field theory.

Newman [4] then conjectured the complementary bound $\Lambda \geq 0$, now known as Newman conjecture, and remarked

> This new conjecture is a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so.

In 2014 J. Stopple [8] generalized Newman's conjecture to other Dirichlet L-functions and gave a lower bound on the analog of Λ. [2] In January 2018 B. Rodgers and T. Tao [7] posted an article to the arXiv proving Newman's conjecture. The generalized version is still open.

3 Li's sequence λ_n

Let

$$\lambda_n := \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left[s^{n-1}\log\xi(s)\right]_{s=1}.$$

In 1997 X.-J. Li [3] proved that the Riemann Hypothesis is equivalent to $\lambda_n \geq 0$ for all $n > 0$. This is known as Li's criterion for the Riemann Hypothesis. Two years latter E. Bombieri and J.C. Lagarias [1] generalized Li's argument showing that it is not specific to the Riemann zeta function. Bombieri and Lagarias [1] also gave arithmetical interpretation for Li's sequence λ_n, via Weil's explicit formula, relating λ_n to prime numbers. It is this arithmetical interpretation that we are going to explore.

[2] Thank you to J. Stopple for the clarification.

4 Some Questions

1. For each $t \in \mathbb{R}$, can we construct a sequence $\lambda_n(t)$ with $\lambda_n(0) = \lambda_n$ such that H_t has only real zeros if and only if $\lambda_n(t) \geq 0$ for all n?
2. Can we find an explicit formula relating a sum over zeros of $H_t(z)$, $0 < t \leq 1/2$, to a sum over primes?
3. Can we adapt the techniques in [7] to settle the generalized Newman conjecture posed in [8]?

4.1 Question 1

The number λ_n can be expressed in terms of over the non-trivial zeros ρ of ζ as

$$\lambda_n = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho} \right)^n \right], \tag{1}$$

where the sum over ρ is understood as

$$\sum_{\rho} = \lim_{T \to \infty} \sum_{|\mathrm{Im}(\rho)| \leq T} . \tag{2}$$

Theorem 1. *([1, Lemma 2]) For $n = 1, 2, 3, \ldots$, the inverse Mellin transform of the function $1 - (1 - 1/s)^n$ is*

$$g_n(x) = \begin{cases} P_n(\log x), & \text{if } 0 < x < 1, \\ n/2, & \text{if } x = 1, \\ 0, & \text{if } x > 1, \end{cases} \tag{3}$$

where $P_n(x)$ is the polynomial

$$P_n(x) = \sum_{j=1}^{n} \binom{n}{j} \frac{x^{j-1}}{(j-1)!}. \tag{4}$$

4.2 Question 2

Let $f(x)$ be a continuously differentiable function on $(0, \infty)$. If

$$F(s) = \int_0^\infty f(x) x^{s-1} dx, \tag{5}$$

then

$$f(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) x^{-s} ds. \tag{6}$$

The function $F(s)$ is called the Mellin transform of $f(x)$. Let ρ be a non-real zero of $\zeta(s)$. Then ρ is also a zero of $\xi(s)$. Assume

$$f(x) = \begin{cases} O(x^\delta) \text{ as } x \to 0^+, \\ O(x^{1-\delta}) \text{ as } x \to \infty, \end{cases} \tag{7}$$

for some $\delta > 0$. Let $\tilde{f}(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$ and $F(s) = \int_0^\infty f(x)x^{s-1}dx$.

Theorem 2. *(Weil explicit formula)*

$$\sum_\rho F(\rho) = \int_0^\infty (f(x) + \tilde{f}(x))dx - (\log \pi + \gamma) f(1) \tag{8}$$

$$- \int_1^\infty (f(x) + \tilde{f}(x) - \frac{2}{x^2} f(1)) \frac{x dx}{x^2 - 1} \tag{9}$$

$$- \sum_{n=1}^\infty \Lambda(n)(f(n) + \tilde{f}(n)). \tag{10}$$

'Explicit' here means that the formula is exact; there is no error term. The main feature of this formula is that the left side is a sum over (non-trivial) zeros of $\zeta(s)$ and the right side involves a sum over primes, hence, connecting zeros of $\zeta(s)$ to primes numbers.

4.3 Question 3

Let $-D < 0$ be a fundamental discriminant and χ the Kronecker symbol. Consider the L-functions

$$L(s, \chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s}. \tag{11}$$

Let

$$\Xi_t(x, \chi) = \int_0^\infty e^{tu^2} \Phi(u, \chi) \cos(ux) du, \tag{12}$$

where

$$\Phi(u, \chi) = 4 \sum_{n=1}^\infty \chi(n) n \exp(3u/2 - n^2 \pi \exp(2u)/D). \tag{13}$$

The function $\Xi_t(x, \chi)$ satisfies the backward heat equation

$$\frac{\partial \Xi}{\partial t} = -\frac{\partial^2 \Xi}{\partial x^2}. \tag{14}$$

In [8] J. Stopple showed that for $t \geq 1/2$, the function $\Xi_t(x, \chi)$ has only real zeros, and if $\Xi_t(x, \chi)$ has only real zeros for some time t, then $\Xi_{t'}(x, \chi)$ also has only real zeros for any $t' > t$. There is a constant $\Lambda_{-D} \in (-\infty, 1/2]$ such that $\Xi_t(x, \chi)$ has only real zeros if and only if $t \geq \Lambda_{-D}$, and it has some complex zeros if $t < \Lambda_{-D}$. Stopple defined

$$\Lambda_{\text{Kr}} = \sup\{\Lambda_{-D} | -D \text{ fundamental}\} \tag{15}$$

generalizing the de Bruijn-Newman constant Λ, then made the conjecture

$$\Lambda_{\text{Kr}} \geq 0, \tag{16}$$

generalizing Newman's conjecture, and gave a lower bound

$$\Lambda_{\text{Kr}} > -1.3 \times 10^{-7} \tag{17}$$

for the analog of the de Bruijn-Newman constant.

References

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