Vinogradov's Mean Value Theorem

Garo Sarajian

University of California, Santa Barbara

4/24/18

Outline

Outline

Introduction

Notation:

- Given a real function $g(x)$ and a function $f(x)$, we have $f(x) = O(g(x))$ means that there exists a constant C independent of x such that $|f(x)| \leq Cg(x)$ for all x.
- $f(x) = o(g(x))$ means $f(x)/g(x) \to 0$ as $x \to \infty$
- We'll also use $\ll$ and $\gg$, where $f(x) \ll g(x)$ is equivalent to $f(x) = O(g(x))$.
- $e(x)$ will be used as shorthand for $e^{2\pi ix}$.
- $s = \sigma + it$ is our complex variable with $\sigma, t \in \mathbb{R}$.

Introduction

Given $k, m \in \mathbb{N}$, and a bound X, we want to count the number of integer solutions $1 \leq x_1, \ldots x_m, y_1, \ldots y_m \leq X$ to the family of k Diophantine equations

$$
\begin{gathered}
x_1 + x_2 + \cdots + x_m = y_1 + y_2 \cdots + y_m \\
x_1^2 + x_2^2 + \cdots + x_m^2 = y_1^2 + y_2^2 \cdots + y_m^2 \\
\vdots \\
x_1^k + x_2^k + \cdots + x_m^k = y_1^k + y_2^k \cdots + y_m^k
\end{gathered}
$$

We denote the number of such solutions as $J_{m,k}(X)$

Introduction

Alternatively, we can say $J_{m,k}(X)$ is the number of integer solutions to

$$\begin{aligned} x_1 + x_2 + \cdots + x_m - y_1 - y_2 \cdots - y_m &= 0 \\ x_1^2 + x_2^2 + \cdots + x_m^2 - y_1^2 - y_2^2 \cdots - y_m^2 &= 0 \\ &\vdots \\ x_1^k + x_2^k + \cdots + x_m^k - y_1^k - y_2^k \cdots - y_m^k &= 0 \end{aligned}$$

with $1 \leq x_j, y_j \leq X$.

As a generalization, we can take a vector $\vec{h} \in \mathbb{Z}^k$ and let $J_{m,k}(X, \vec{h})$ be the number of integer solutions to

$$\begin{aligned} x_1 + x_2 + \cdots + x_m - y_1 - y_2 \cdots - y_m &= h_1 \\ &\vdots \\ x_1^k + x_2^k + \cdots + x_m^k - y_1^k - y_2^k \cdots - y_m^k &= h_k \end{aligned}$$

with $1 \leq x_j, y_j \leq X$.

Introduction

Let

$$f_k(\vec{\alpha}, X) = \sum_{1 \leq x_1, x_2, \ldots x_k \leq X} e(\alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_k x^k)$$

Then

$$J_{s,k}(X) = \int_{[0,1]^k} |f_k(\vec{\alpha}, X)|^{2m} \, d\vec{\alpha}$$

Also,

$$J_{s,k}(X, \vec{h}) = \int_{[0,1]^k} |f_k(\vec{\alpha}, X)|^{2m} e(-h_1\alpha_1 - \cdots - h_k\alpha_k) d\vec{\alpha}$$

In particular, $J_{m,k}(X) \geq J_{m,k}(X, \vec{h})$ for any $\vec{h}$.

Introduction

Looking over all possible solutions values of $x_j, y_j \in [1, X]$, we get

$$\sum_{\vec{h}\in\mathbb{Z}} J_{m,k}(X,\vec{h}) \leq J_{m,k}(X) \sum_{\substack{\vec{h}\in\mathbb{Z} \\ J_{m,k}(X,\vec{h})>0}} 1$$

The left hand side is X^{2m}, and the sum on the right hand side is $\ll X \cdot X^2 \cdots X^k = X^{k(k+1)/2}$.
So

$$X^{2m} \ll J_{m,k}(X) X^{k(k+1)/2}$$

Rearranging, we get a lower bound:

$$J_{m,k}(X) \gg X^{2m - \frac{k(k+1)}{2}}$$

Introduction

Diagonal Solutions: Taking $x_j = y_j$, we see that there are X^m diagonal solutions, so $J_{m,k}(X) \geq m! X^s \gg X^m$.
Combining our two lower bounds for $J_{m,k}(X)$, we get

$$J_{m,k}(X) \gg \max\left\{X^m, X^{2m - \frac{k(k+1)}{2}}\right\}$$

$$J_{m,k}(X) \gg X^m + X^{2m - \frac{k(k+1)}{2}}$$

Introduction

Vinogradov's Mean Value Theorem is used to refer to any nontrivial upper bound on $J_{m,k}(X)$.
Theorem [Wooley, Bourgain, Demeter, Guth, 2016]:

$$J_{m,k}(X) \ll X^{m+\epsilon} + X^{2m-\frac{k(k+1)}{2}+\epsilon}$$

Outline

Relationship with ζ

Recall

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

If

$$\Lambda(n) = \begin{cases} \log(p) & : n = p^a \\ 0 & : (pq)|n \end{cases}$$

then

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Relationship with ζ

Prime Number Theorem [de la Vallée Poussin, Hadamard 1896]

$$\sum_{p \leq x} 1 \sim \frac{x}{\log(x)}$$

Proof:

$$\begin{aligned}\text{Prime Number Theorem} &\Leftrightarrow \sum_{n \leq x} \Lambda(n) = x + o(x) \\ &\Leftrightarrow \zeta(1 + it) \neq 0\end{aligned}$$

Relationship with ζ

Proof (cont): For $\sigma \geq 1$, we have

$$
\begin{aligned}
&-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} - Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} \\
&= \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} \left(3 + 4\cos(t\log(n)) + \cos(2t\log(n))\right) \\
&= 2\sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} \left(1 + \cos(t\log(n))\right)^2 > 0
\end{aligned}
$$

Relationship with ζ

- In general, better bounds on the zero-free region of ζ means better bounds on the error term in the Prime Number Theorem $\sum_{n\leq x} \Lambda(n) = x + o(x)$.
- Riemann Hypothesis (all non-trivial zeros s have $Re(s) = 1/2$) $\Leftrightarrow \sum_{n\leq x} \Lambda(n) = x + O\left(x^{1/2}\log^2(x)\right)$.
- Vinogradov's Mean Value Theorem $\Rightarrow$ bounds on exponential sums $\Rightarrow$ bounds on $\zeta(s)$ in a certain region $\Rightarrow$ wider zero-free region.
- Specifically, $\zeta(\beta + it) \neq 0$ if $\beta \geq 1 - (\log(t))^{-2/3} (\log(\log(t))^{-1/3}/57.54$, and

$$\sum_{n\leq x} \Lambda(n) = x + O\left(xe^{-\log^{3/5}(x)/(57.54\log(\log(x)))^{1/5}}\right)$$

Outline

Relationship with Waring's Problem

- Waring's problem asks how many kth powers are needed to represent natural numbers. It's basically solved when we're trying to represent all natural numbers, but when trying to represent all sufficiently large numbers, much less is known.
- Define $G(k)$ as the smallest number a such that all sufficiently large numbers can be written as a sum of a k-th powers.
- Example: Lagrange's Four Square Theorem says that all positive numbers can be written as a sum of four squares, so $G(2) \leq 4$. Since $8n + 7$ requires four squares, $G(2) = 4$.
- $G(k)$ is known only for $k = 1, 2, 4$, ($G(4) = 16$). Vinogradov's Mean Value Theorem allow us to lower bounds for $G(k)$.

Relationship with Waring's Problem

- Much of the progress on Waring's problem comes from limiting the number of solutions to

$$x_1^k + \cdots + x_s^k = y_1^k + \cdots + y_s^k$$

where $1 \leq x_i, y_i \leq X$.

- The number of such solutions is $\ll J_{s,k}(X)X^{k(k-1)/2}$. The proof of Vinogradov's Mean Value Theorem has improved the bounds for $G(k)$ for most values of k: $G(7) \leq 31$, $G(8) \leq 39$, $G(9) \leq 47 \ldots$

- J. Bourgain, C. Demeter, and L. Guth. Proof of the main conjecture in Vinogradov?s mean value theorem for degrees higher than three. Ann. of Math. (2), 184(2):633-682, 2016.
- N. M. Korobov, Estimates of trigonometric sums and their applications, Uspehi Mat. Sauk 13 (1958), 185-192. (Russian)
- T. Tao. 254A, “Notes 2: Complex-analytic multiplicative number theory." Web blog post. What’s New. 2014
- R. C. Vaughan. The Hardy-Littlewood Method, volume 125 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, New York, 2nd edition, 1997.
- I. M. Vinogradov. A new estimate for $\zeta(1+it)$, Izv. Akad Nauk SSSR, Ser. Met. 22 (1958), 161-164, (Russian)
- T. D. Wooley. On Waring’s problem for intermediate powers. Act Arith., 176(3): 241-247, 2016

T. D. Wooley. Vinogradov's mean value theorem via efficient congruencing. Annals of Math. 175 (2012), 1575–1627.