A Diagrammatic Approach to Skein Algebras

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The Kauffman bracket of a link gives a way of defining the Jones polynomial entirely through combinatorics on link diagrams. The key to this approach is to describe a way to resolve any crossings that appear in the diagram, until only a finite number of loops remain. Diagrammatically this allows us to define the following relations on any link diagrams:

$$\times = A \asymp - A^{-1} \;)(\tag{1}$$

and

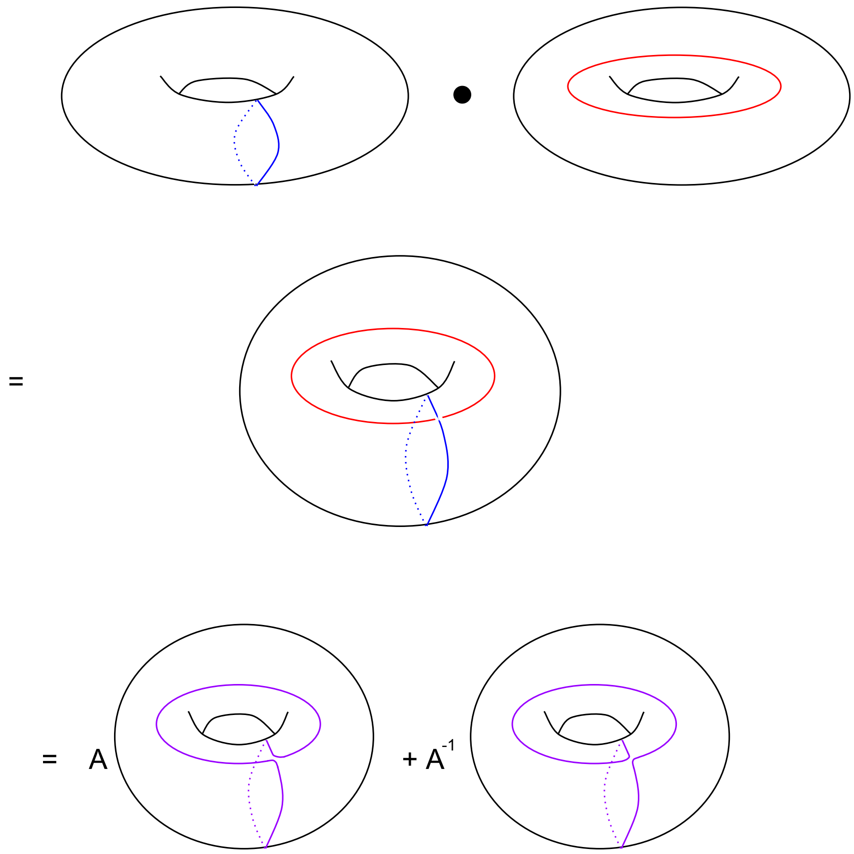
$$\bigcirc \cup L = -(A^2 + A^{-2})L, \tag{2}$$

where the framed links being described are identical outside of a ball containing these diagrams. This process can be carried out to give a Laurent polynomial in the variable A to any link. As a generalization of this process an invariant of a $3-$manifold M can be introduced as the Kauffman bracket skein module of M, denoted $K_A(M)$. We define

$$K_A(M) := \mathbb{C}\mathcal{L}/\sim$$

to be the free complex vector space generated by $\mathcal{L}$, where $\mathcal{L}$ is the set of all isotopy classes of links in M, and the equivalence relation $\sim$ comes from the Kauffman bracket relations described above. In the specific case where $M = \Sigma \times [0,1]$ with Σ a surface, we are able to look at $K_A(\Sigma \times [0,1])$ as an algebra. The multiplication which we are giving this vector space is simply stacking, meaning $\alpha \cdot \beta$ is the skein obtained by staking α over β and then resolving.

As an example we can look at the multiplication of two curves in $K_A(\mathbb{T}^2)$:



This skein theoretic formalism has provided an elementary approach to many results involving much deeper mathematics. In particular this algebra, when specializing the variable A to specific roots of unity is of great interest. This project in particular will involve using combinatorics and other methods to work in this diagrammatic algebra to explore properties and build a better understanding for varying surfaces. A background in knot theory will be useful, but not necessary. A desire to learn and a willingness to work with pictorial arguments are key for this project.