

Analysis in almost Abelian homogeneous spaces

Zhirayr Avetisyan (UCSB)

www.z-avetisyan.com

Prerequisites: The present research project touches upon many advanced mathematical subjects, but no prior knowledge is required to participate in this programme except:

- Undergraduate level linear algebra (e.g., real and complex vector spaces and operators, operations with matrices)
- Undergraduate level analysis (e.g., continuous and differentiable functions of several variables, normed vector spaces)
- Strong problem solving skills
- Acquaintance with standards of mathematical writing (e.g., definitions, proofs)
- Modesty and ambition :-)

Introduction: In any vector calculus class (lower division) students learn about functions defined on surfaces $\Sigma \subset \mathbb{R}^3$ embedded in the three-space \mathbb{R}^3 . They learn how to differentiate and integrate such functions using various parameterizations of Σ , and are (hopefully) cautioned that not every surface can be entirely covered by a single parameterization. This brings us to the first mathematical subject concerned in this project - *differential geometry*. This is, roughly speaking, the study of often higher dimensional "hyper-surfaces" Σ (called *manifolds*) which can be covered by a collection of mutually smoothly agreeing smooth parameterizations. In many cases the manifold Σ is defined through a system of algebraic equations relating the Cartesian coordinates x_1, \dots, x_n of the ambient space \mathbb{R}^n , $n \in \mathbb{N}$ (think of the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ defined by $x_1^2 + x_2^2 + x_3^2 = 1$). Here we very softly touch another mathematical subject - *algebraic geometry*, which, however, will not play a central role in this project. Together we will study a class of higher dimensional manifolds defined by algebraic equations on matrices - matrix equations, and we will consider functions and vector fields defined on such manifolds.

Linear algebra defines the operation of multiplication for matrices, which is shown to be associative (i.e., $(AB)C = A(BC)$). A *matrix group* is a set G of $n \times n$ matrices such that:

- $1 \in G$ (identity matrix is in G)
- $\forall A, B \in G, C = AB \in G$ (G is closed under matrix multiplication)
- $\forall A \in G, \exists A^{-1} \in G$ (all elements of G are invertible, and inverses also lie in G)

Almost Abelian groups are matrix groups which have a very special structure. The algebraic equations defining our manifolds Σ discussed above will involve algebraic operations with almost Abelian groups G , and this interaction with G will completely determine the properties of the manifold Σ , which will therefore be referred to as an almost Abelian homogeneous space. One of our main tasks will be to study these properties for a given almost Abelian group G and a suitable manifold Σ .

Another notion from a vector calculus class is that of tangent vectors \vec{T} to a surface Σ . A tangent vector $\vec{T}(x)$ at a point $x \in \Sigma$ can be thought of as the tangent vector $\vec{T}(x) = \gamma'(0)$ to a smooth curve $\gamma : [0, 1] \rightarrow \Sigma$ with $\gamma(0) = x$. If we think of a matrix group G as a manifold $\Sigma = G$ then the identity matrix $x_0 = 1$, as a point on the manifold, has a very special role. Tangent vectors to $\Sigma = G$ at $x_0 = 1$ form an interesting algebraic object called a *Lie algebra*, which essentially encodes almost all information about G . In fact, every tangent vector $\vec{T}(x_0)$ as above can be represented by a matrix X_T . Moreover, if we take the matrix exponential (in terms of Taylor's expansion) of $t \cdot X_T$ for various numbers $t \in \mathbb{R}$,

$$e^{t \cdot X_T} = 1 + t \cdot X_T + \frac{1}{2}t^2 \cdot X_T^2 + \dots,$$

then we obtain matrices which are elements of the matrix group G . It turns out that most elements of G can be recovered in this way, but not necessarily all. Finding this out will be one of our tasks in this project, and in order to do so we need to understand well the convergence of the above Taylor's series in various matrix norms.

Discussion: Of course, it is not expected that every student acquires a good understanding of all these subjects in the short time frame of eight weeks. There will be three main regimes of learning new material:

- Independent reading (starting well before the first day of the programme)
- Crash courses on specific topics during the programme
- Learn as you go (learning by doing)

The mathematical problems to be addressed are of highest scientific standards, and solving them will require much work and utmost devotion (and, obviously, talent). Students will be provided all the guidance needed, but will also be expected to be autonomous. The research output will be published in prestigious international journals and/or as part of a future monograph, and the participating students will act as coauthors.

Applications: This project is in pure mathematics, and its most direct applications are also of theoretical nature. Remember that Einstein's General Relativity theory remained as an essentially purely theoretical mind practice for almost a century before the humankind came up with a profound application of it, which shapes our daily lives today - the GPS.

One of the immediate applications of analysis on almost Abelian spaces is mathematical cosmology. Supported by many observations, it is believed that our universe is *homogeneous* on the large scale; this means that the large scale structure of the universe (e.g., the density of clusters of galaxies) looks fairly identical from the perspectives of all hypothetical observers in different parts of the space. The present day universe is also considered to be *isotropic*, i.e., it looks the same in all directions if you average out "small" local deviations (such as an individual galaxy). However, the universe need not have been isotropic in its distant and violent past - some minutes after the hypothetical Big Bang, say. Building rigorous mathematical methods for modelling the behaviour of early universe is the main objective of mathematical cosmology. If we model the space by a manifold Σ , then the properties of homogeneity or isotropy can be expressed in terms of geometrical symmetries. It turns out that, if we adopt the 4-dimensional space-time paradigm of General Relativity, then most homogeneous but non-isotropic models of early universe have their geometrical symmetries in form of 3-dimensional almost Abelian groups. Every advancement in our understanding of mathematics on almost Abelian spaces leads to innovations in mathematical cosmology (which participating students may also choose to partake in, if they wish).

Literature: References for the mathematical background mentioned above will be provided to students in the due course. Some recent work on almost Abelian structures can be found in the following preprints:

- Z. Avetisyan, "Structure of almost Abelian Lie algebras", arXiv:1610.05365, 2016
- Z. Avetisyan, "Jordanable almost Abelian Lie algebras", arXiv:1811.01252, 2018

For an instance of application in mathematical cosmology:

- Z. Avetisyan and R. Verch, "Explicit harmonic and spectral analysis in Bianchi I-VII type cosmologies", *Classical and Quantum Gravity*, 30 (15), 2013