Analysis in almost Abelian homogeneous spaces

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Prerequisites: The present research project touches upon many advanced mathematical subjects, but no prior knowledge is required to participate in this programme except:

- Undergraduate level linear algebra (e.g., real and complex vector spaces and operators, operations with matrices)
- Undergraduate level analysis (e.g., differential and integral calculus of functions of several variables, vector fields)
- Strong problem solving skills
- Acquaintance with standards of mathematical writing (e.g., definitions, proofs)
- Modesty and ambition :-)

Introduction: In any vector calculus class (lower division) students learn about functions defined on surfaces $\Sigma \subset \mathbb{R}^3$ embedded in the three-space $\mathbb{R}^3$. They learn how to differentiate and integrate such functions using various parameterizations of Σ, and are (hopefully) cautioned that not every surface can be entirely covered by a single parameterization. This brings us to the first mathematical subject concerned in this project - *differential geometry*. This is, roughly speaking, the study of often higher dimensional "hypersurfaces" Σ (called *manifolds*) which can be covered by a collection of mutually smoothly agreeing smooth parameterizations. In many cases the manifold Σ is defined through a system of algebraic equations relating the Cartesian coordinates $x_1, \ldots, x_n$ of the ambient space $\mathbb{R}^n$, $n \in \mathbb{N}$ (think of the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ defined by $x_1^2 + x_2^2 + x_3^2 = 1$). Here we very softly touch another mathematical subject - *algebraic geometry*, which, however, will not play a central role in this project. Together we will study a class of higher dimensional manifolds defined by algebraic equations on matrices - matrix equations, and we will consider functions and vector fields defined on such manifolds.

One very important notion from a vector calculus class is that of tangent vectors $\vec{T}$ to a surface Σ. A tangent vector $\vec{T}(x)$ at a point $x \in \Sigma$ can be thought of as the tangent vector $\vec{T}(x) = \gamma'(0)$ to a smooth curve $\gamma : [0,1] \to \Sigma$ with $\gamma(0) = x$. If a vector $\vec{T}(x)$ is defined for every $x \in \Sigma$ and $\vec{T}(x)$ varies smoothly with x then $\vec{T}$ describes a smooth vector field on Σ. Furthermore, the set of all vectors $\vec{T}(x)$ at a fixed point x comprises a vector space called the tangent space $\mathrm{T}_x\Sigma$ of Σ at x, and tensor products of copies of $\mathrm{T}_x\Sigma$ and its linear dual space $\mathrm{T}_x^*\Sigma$ contain all tensors at x. Again, if a tensor is prescribed at every $x \in \Sigma$ and that prescription is smooth then it defines what is called a smooth tensor field on Σ.

Linear algebra defines the operation of multiplication for matrices, which is shown to be associative (i.e., $(AB)C = A(BC)$). A *matrix group* is a set G of $n \times n$ matrices such that:

- $1 \in G$ (identity matrix is in G)
- $\forall A, B \in G$, $C = AB \in G$ (G is closed under matrix multiplication)
- $\forall A \in G$, $\exists A^{-1} \in G$ (all elements of G are invertible, and inverses also lie in G)

(One often adds the requirement of G being closed as a subset of $\mathrm{GL}(n)$.) Almost Abelian groups are matrix groups which have a very special structure. If we think of an element $x \in G$ of a matrix group as a point and of matrix entries as coordinates, then we can consider the group G as our hypersurface $\Sigma = G$. For every fixed matrix $A \in G$, the map $\Sigma \ni x \mapsto Ax \in \Sigma$ defines a global transformation, which affects all vector and tensor fields defined on Σ. A vector (tensor) field is left-invariant if it remains the same after the transformation caused by every $A \in G$ as above. In this project we will study several important geometrical structures on $\Sigma = G$ which are left-invariant, such as contact and symplectic structures.

Discussion: Of course, it is not expected that every student acquires a good understanding of all these subjects in the short time frame of eight weeks. There will be three main regimes of learning new material:

- Independent reading (starting well before the first day of the programme)
- Crash courses on specific topics during the programme
- Learn as you go (learning by doing)

The mathematical problems to be addressed are of highest scientific standards, and solving them will require much work and utmost devotion (and, obviously, talent). Students will be provided all the guidance needed, but will also be expected to be autonomous. The research output will be published in prestigious international journals and/or as part of a future monograph, and the participating students will act as coauthors.

Applications: This project is in pure mathematics, and its most direct applications are also of theoretical nature. Remember that Einstein's General Relativity theory remained as an essentially purely theoretical mind practice for almost a century before the humankind came up with a profound application of it, which shapes our daily lives today - the GPS.

One of the immediate applications of analysis on almost Abelian spaces is mathematical cosmology. Supported by many observations, it is believed that our universe is *homogeneous* on the large scale; this means that the large scale structure of the universe (e.g., the density of clusters of galaxies) looks fairly identical from the perspectives of all hypothetical observers in different parts of the space. The present day universe is also considered to be *isotropic*, i.e., it looks the same in all directions if you average out "small" local deviations (such as an individual galaxy). However, the universe need not have been isotropic in its distant and violent past - some minutes after the hypothetical Big Bang, say. Building rigorous mathematical methods for modelling the behaviour of early universe is the main objective of mathematical cosmology. If we model the space by a manifold Σ, then the properties of homogeneity or isotropy can be expressed in terms of geometrical symmetries. It turns out that, if we adopt the 4-dimensional space-time paradigm of General Relativity, then most homogeneous but non-isotropic models of early universe have their geometrical symmetries in form of 3-dimensional almost Abelian groups. Every advancement in our understanding of mathematics on almost Abelian spaces leads to innovations in mathematical cosmology.

Literature: References for the mathematical background mentioned above will be provided to students in the due course. Some recent work on alsmot Abelian structures can be found in the following preprints:

- Z. Avetisyan, "Structure of almost Abelian Lie algebras", arXiv:1610.05365, 2016
- Z. Avetisyan, "Jordanable almost Abelian Lie algebras", arXiv:1811.01252, 2018

For an instance of application in mathematical cosmology:

- Z. Avetisyan and R. Verch, "Explicit harmonic and spectral analysis in Bianchi I-VII type cosmologies", Classical and Quantum Gravity, 30 (15), 2013