Abstract: The purpose of this paper is to present results from an ongoing design research study that explores the development of early pedagogical content knowledge (PCK). Prospective elementary and secondary teachers participated in this study while enrolled in mathematics courses designed to 1) strengthen their own understanding of K-12 mathematics, 2) enhance their understanding of the strategies and models that children use in mathematics problem solving and 3) develop their capacity to be able to see the development of the big mathematical ideas in children’s work and to analyze subsequent instruction that is based on that development. These aspects of understanding of children’s thinking—critical components of PCK—were measured throughout the courses using paper and pencil measures of MPCK and participants’ reflections on video cases of students working on investigations. Results of analyses of data collected from two administrations of these measures are used to compare the differences in understandings of children’s mathematics between these two groups of prospective teachers. Implications for including development of critical aspects of understanding of children’s thinking in teacher education are discussed.

Introduction
The use of video cases in teacher education has a long history that has changed in recent years from having teachers watch snip-bits of instruction designed to illustrate appropriate teaching techniques to accessing complex database of classroom activities with accompanying materials designed to have teachers analyze student thinking, reflecting on purposes of instruction and learn important mathematical content. A recent study by Santagata (2009) designed to report how video cases were used to develop the pedagogical content knowledge of practicing teachers in low performing schools, brought up important issues regarding the gaps that the practicing teachers have in their PCK. Not only did these teachers have difficulties understanding their children’s knowledge of important mathematical concepts and analyzing their student’s work on any level other than judging it right and wrong but their own understandings of core concepts that were to be taught needed to be developed. In this paper we illustrate how these important aspects of Mathematical PCK (MPCK)—namely understanding of children’s thinking and ability to see and understand the big ideas in mathematics, are developed early in mathematics for prospective teachers, while these future teachers are still in undergraduates. In these courses in which video cases are used, participants develop understandings of children’s mathematics while exploring for themselves the big ideas in the k-12 mathematics curriculum. We use an analysis of participant’s responses to prompts embedded in the video cases themselves and their own mathematical investigations as well as more typical measures of MPCK (Hill & Ball, 2007) to compare important differences in the early MPCK found in prospective elementary and secondary teachers. In doing so we contribute to the mathematics education research literature in what Hill and colleagues refer to as an underspecified and under-demonstrated domain (Hill, et. al, 2008, p2). These differences have implications for not only the education of prospective teachers but the professional development of practicing teachers.

1 This research was supported by the National Science Foundation and the Educational Advancement Foundation.
Mathematical Pedagogical Content Knowledge

PCK, a construct first proposed by Lee Shulman two decades ago (Shulman, 1986, 1987), and revisited since by others in various content fields (Ball and Bass; 2000, Morine-Dershimer and Kent, 2001 and Mishra & Koehler, 2006), is that special amalgamation of content and pedagogical knowledge a good teacher draws on to make instructional decisions. Recent work into Mathematical Pedagogical Content Knowledge (MPCK) has explored the specific types of knowledge needed by a teacher in order to teach mathematics for understanding (Hill & Ball, 2004). MPCK includes: 1) a deep and connected understanding of mathematics concepts, 2) an understanding of appropriate pedagogical approaches, and 3) an understanding of children’s mathematics, including the propensity to see and assess the big mathematical ideas in children’s mathematics and to base subsequent instruction upon that knowledge. It is (1) and (3) and the important connections between them that we focus on in this paper.

Recent Literature on Using Children’s Thinking In Teacher Development

Recent literature that researches the results of professional development workshops or programs that focus on an understanding of children’s thinking have presented results about the how this perspective has changed elementary teachers’ beliefs and understanding of mathematics content (Philipp et. al., 2007, Feikes, Pratt & Hough, 2006) It has been suggested that this deepening understanding of children’s mathematics may lead to a teacher changing their practice. Indeed, Steinberg and colleagues (2004) describe the dramatic change one teacher had during her third year of teaching by learning to listen to her children about their thinking during mathematics instruction. In their investigation of teachers’ mathematical knowledge, (Empson &Junk 2004) found that teachers’ greatest source of mathematical knowledge was from the children that they taught and Philipp and his colleagues (Philipp, et.al, 2003, 2007) found that in addition to enhancing prospective teachers beliefs about mathematics, using a children’s thinking approach acted as the motivator for them to learn more mathematics. Phillip and colleagues (2007) also found that elementary teachers that used video case study technology to learn about children’s thinking along with their own explorations of mathematics changed their beliefs more deeply than those elementary teacher who had apprenticeship experiences under carefully selected teachers in live classrooms. In this paper we add to these findings by discussing both how these two aspects of mathematical knowledge are intrinsically linked. We further illustrate important differences between the ways in which prospective elementary and secondary teachers develop in their abilities for understanding children’s mathematical thinking when video case study technology is used and how these differences are connected to the ways in which these groups approach mathematical investigations in general rather than the amount of content that they have. We suggest that these differences are relevant to the issues of content knowledge deficits in practicing teachers (ie. Santagano, 2007).

Theoretical Framework

Anyone who has even been the least interested in mathematics or has even observed other people who were interested in it, is aware that mathematical work is work with ideas. Symbols are used as aids to thinking just as musical scores are used to aid musicians. The music comes first the scores come later.

Hersh (1986).

The theory that guides our work draws from constructivism (Glaserfeld, 1989) as an epistemology and the didactic/realistic mathematics (Fosnot & Dolt, 2001) as a pedagogical stance. Within this perspective, knowledge is seen as an adaptive function in which a learner comes to “know” by constructing their own knowledge as they respond to meaningful, well-crafted problem contexts that are chosen to bring forth big ideas in the mathematics curriculum. As such, in this study, instructional sequences are designed for participating prospective teachers to develop understandings of mathematical thinking of students in K-12 classrooms. These
sequences include video cases of children investigating open ended problems that are part of carefully crafted instructional sequences in which they investigate big ideas at, and just beyond, their particular developmental levels. We use the emergent strategies, big ideas and models found in children’s inquiry and use them as an organizing framework in order to look for pre and in-service teachers to understand their own mathematizing and develop their abilities to describe the mathematizing of children. The task for the prospective teacher is to make meaning out of the children’s understandings in order to analyze instructional mathematical decisions appropriate for the children they are teaching. For these reasons we call the work that the prospective teachers do “investigations on investigations.”

**Methodology**

Nineteen prospective elementary and nineteen prospective secondary teachers participated in this study while enrolled in 2-quarter sequences of mathematics courses (Math 100AB and Math 181AB respectively). This study is part of a larger evaluation study being conducted as a design study (Cobb et al, 2001) to explore the development of MPCK over time. It is in this larger context that two different understandings of children’s thinking emerged that were of interest in looking at said development: (1) being able to understand and describe the strategies and models that children construct when exploring mathematics; (2) seeing the developmental nature/big ideas in children’s work with prospective elementary teachers (PETs) more adapt at these than the prospective secondary teachers (PSTs). We hypothesized that these differences were connected in complex ways to (3) the mathematical classroom practices and individual interpretations of mathematics of the two groups (Cobb, et. al. 2001). In order to explore this hypothesis we devised an instructional sequence in which we could explore (1), (2) and (3) above. This sequence consisted of small groups of students investigating a rich mathematical problem a small group and whole class investigation on investigation of a video case study of a related problem. In this investigation prospective teachers were introduced to a 4th grade context, explored the problem themselves and then investigated the children’s investigation of the context. To allow us to understand how individual prospective teachers took up this work, in the final piece of the sequence individuals investigated children’s mathematics through a second video case in another related context.

**Data Collection**

Two video cameras were used in each classroom to capture both the whole class discussions and to capture two groups of student in each classroom as they worked. Written work was collected from the individual problem for analysis. From this sequence we were able to understand prospective teachers group problem solving activities which were then shared whole class; how groups of prospective teachers transferred knowledge of one problem to another, how groups of prospective teachers and whole classes interpreted children exploring the problem and how individuals transferred these understandings in order to investigate children working in a second related context.

**Instructional Sequence**

Both groups of prospective teachers were asked to:

1. Investigate how many rectangular solids could be found with a lateral surface area of 24 square units. This investigation was set in the context of a person working on a farm binding stacks of hay bales around the sides in order to stop the wind from blowing the hay around. Prospective teachers worked in groups and were given unifix cubes to build the hay stacks if they chose and 24 (unit) strip of paper that represented chicken wire with which to wrap the bales. Several big ideas were embedded in this context. The first is the understanding that rectangular solids with
various volumes can give the same lateral surface area. Another is the meaning of “similar rectangular solids” in this context.

2. View clips of a related context being implemented in a 4th grade elementary class (Mikki’s Chocolate Box Problem) and were asked to analyze how this context related to big mathematical ideas in the 6th grade curriculum. The teacher asked her class how many different boxes that hold 24 chocolates can be found if each chocolate takes up one square unit of space. This problem context introduces a 3 dimensional array as a model—the teacher brought in a 2 layer chocolate box that consisted in which each layer had 12 chocolates arranged in 3 rows of 4. This context gave children an opportunity to explore the associative and commutative properties, of multiplication, factor pairs, doubling and halving strategies, and ways of proving that they had found all possible boxes.

3. Work in groups to solve the problem that had been posed to the children. Whole class discussions took place to analyze different solution strategies.

4. Work in groups to investigate the 4th grade children’s mathematical thinking as they solved the Chocolate Box Problem. Whole class discussion then took place to describe the different solutions that children used and their relationship to the big mathematical ideas in their work.

5. Investigate, on their final exam, the same 4th grade children’s mathematical thinking on a follow-up problem in which children investigated the price of to make the chocolate boxes that they had found in their previous investigation. Children were asked to find the most and least expensive boxes. This would be the first time that the children investigate surface area and its relationship to volume.

Video data and written work from this sequence was coded using constant comparative techniques of Glaser (1992). In addition, we administered the Mathematical Knowledge for Teaching Survey (Hill, Schilling & Ball, 1994) pre course as well as a administering a four-item researcher developed questionnaire with questions modified from the MKT that asked participants to explain their answers. Frequencies from items on the MKT, were compared between PSTs and PET. Open-ended responses on the researcher-developed questionnaire were coded using constant comparative techniques.

Results
Results indicate that while there were no significant pre differences on the MKT nor in the number of correct answers to the constructed response items that resembled the MKT items (full results of data tables illustrating this will be included in the presentation and final paper) the differences between responses to the video sequences were profound as were open-ended responses to the constructed items. See Table 1 below for the categories and sub categories that emerged connected to the PTs development of understanding of children’s thinking as evidenced on 4) of the instructional sequence above.

Insert Table 1 here

While the two groups are comparable in both their abilities to describe the strategies that children use to do mathematics and their understanding of how contexts are related to big mathematical ideas the groups differed in their ability to see the big ideas in children’s work and perhaps more importantly see the development of those ideas and the links between them with twice as many prospective teachers as P prospective teachers able to do this. To explore connects between this development and the ways in which the mathematical classroom practices had emerged between the two groups of prospective teachers we coded their own investigations in mathematics as evidenced in the Maggie’s Farm problem and the Chocolate box problem (see Table 2 below).
The main differences in the classroom mathematical practices of the PETs and the PSTs is the ways in which they approached and investigated the problem. The PET’s all grounded their investigations in the context of the haystacks which allowed them to uncover important ideas about the relationships between dimensions and the meaning of congruence when thinking of lateral surface area. Three of the four groups developed a constraint for the dimensions of the haystacks in this problem as they worked. The prospective teachers on the other hand started by generating a constraint (which became the model) and then testing various values of the dimensions in order to find all possible solutions. In this particular case the choice of height to fix was not made because of the context of the problem and due to the emergent 3d array model that fosters this choice, but because of its placement in the formulaic model, i.e. they had constructed \((l+w)\) as an entity and \(h\) as an entity so the latter was the obvious choice for fixing. Due to this, the question of whether flipping the rectangular prism by exchanging \(l\) and \(h\) or \(w\) and \(h\) did not arise. Nor were any of the relationships between the dimensions explored. Specific examples from classroom vignettes of PETs and PSTs will be included in the presentation and the final paper to illustrate these differences.

Table 3 below offers a similar chart of the classroom mathematical practices that occurred during the chocolate box problem (the problem that was investigated by the children). The first thing that stands out is the transfer of the layering strategy for all groups that used a context driven model in Maggie’s farm and the lack of transfer of the strategy of fixing the height in the Chocolate Box.

As can be seen in table 3 above, all four groups of PETs generalized the strategy that they used in Maggie’s Farm to use in Mikki’s Chocolate Box whereas only the one group of PSTs that worked in context did. All of the other four groups saw the Chocolate Box as a problem involving a separate strategy. This led them 3 of the remaining 4 groups into trouble as they developed their non context based strategy of fixing \(l\) the length in their model/formula \(lwh = 24\). All of the PST’s in these groups ran into cognitive difficulties when thinking about what congruent rectangular prisms meant in this case due to the fact that they had not considered it in the previous problem and the added confound of their strategy of looking at slices of a rectangular prism rather than layers of a chocolate box. Specific vignettes from both small group and whole class discussion will be given in the final paper to illustrate what occurred. The result of these differing orientations to the problem resulted in the PETs generating all of the solutions in a third of the time that the PSTs worked on the problem without satisfactory resolution (the instructor had to intervene and lead the solution).

The differences in the ways in which the two groups of prospective teachers explored the mathematics also served to constrain the “noticing” of the k-12 big ideas that were implicit in the context. These differences could be a factor in explaining the differences found in the ways that the groups were able to understand the development of big ideas in the children’s work. Figures 1 and 2 below trace the trajectories of the PETs and the PSTs as they participated in the sequence of activities leading up to the final examination and shows the connection between their small group mathematical practices and their ability to interpret the children’s mathematics.

**Implications**

The paper and pencil measures of MPCK include items that measure only two aspects of understanding of children’s thinking/or mathematizing—the ability to understand the strategies that students use and the ability to pick up on the cognitive obstacles that children typically have. The important facility to be able to see the big ideas in children’s work and more critically the
development of those ideas is much harder to both develop and measure. Using classroom video analysis and prospective teachers’ own responses to video cases gave us insight into the ways in which these prospective teachers were able to understand aspects of children’s mathematical thinking. We found that the ways in which prospective teachers are able to mathematize will serve to enhance or constrain the ways in which they can understand and interpret children’s mathematics. Prospective secondary teachers with more mathematical experience, most probably in traditional settings, has given them a procedural orientation to mathematics that is hard to overcome. Prospective elementary teachers on the other hand were much more able to change their ways of looking at mathematics and mathematizing, embracing the latter and by doing so gave them more opportunities for exploring first hand the types or mathematical relationships explored by the children.

References


Santagata, R. (2009). Designing Video-Based Professional Development for Mathematics Teachers in Low-Performing Schools. *Journal of Teacher Education*, 60(1), 38-?

### Table 1. Prospective Teachers’ Understanding of Children’s Thinking

**Understanding the context and the models that are inherent in them.**

<table>
<thead>
<tr>
<th>Level</th>
<th># of PETs</th>
<th># of PSTs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0: 0 Doesn’t mention context</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>Level 1: Explains context as “interesting problem” or in other ways misinterprets what context is.</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Level 2: Tries to describe context in terms of the mathematical ideas or strategies that it may bring forth but misses important points or is confused.</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Level 3: Describes the context and its role in bringing forth big ideas and strategies/models.</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

**Describing the strategies that children use.**

<table>
<thead>
<tr>
<th>Level</th>
<th># of PETs</th>
<th># of PSTs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0: Doesn’t mention or gives a general description not related to individual student work.</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Level 1: Repeats what the student says or misinterprets what the student is doing.</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Level 2: Correctly interprets and describes some of the strategies but incomplete picture or missing analysis</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Level 3: Finds evidence and describes the strategies and connects to the big ideas in student work.</td>
<td>14</td>
<td>14</td>
</tr>
</tbody>
</table>

**Seeing the development of big mathematical ideas in children’s work.**

<table>
<thead>
<tr>
<th>Level</th>
<th># of PETs</th>
<th># of PSTs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0: Sees the congress in Miki’s second congress as a share out.</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>Level 1: Identifies progression of strategies in Miki’s second congress but misses big ideas</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Level 2: Identifies big ideas in the children’s work and sees some connections between the development of those ideas</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>Level 3: See big ideas in the children’s work and is able to see the development of those ideas.</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Level 4: Identifies Miki’s intervention as raising relationship between halving and doubling and the overlap in reducing surface area.</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 2. Prospective Teachers' Investigation of Maggie’s Farm Problem

<table>
<thead>
<tr>
<th>Maggie’s Farm</th>
<th>Strategies observed/Models that emerged/Big ideas discovered</th>
<th>PETs</th>
<th>PSTs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Building with blocks and making lists of possible haystacks</td>
<td>4/4 groups</td>
<td>1/5 groups</td>
<td></td>
</tr>
<tr>
<td>Focusing on a fixed number of layers, building and making lists for that layer number</td>
<td>4/4 groups</td>
<td>1/5 groups</td>
<td></td>
</tr>
<tr>
<td>Using a compensation strategy to generate all of the possible dimensions for L and W within a fixed layer</td>
<td>4/4 groups</td>
<td>0/5 groups</td>
<td></td>
</tr>
<tr>
<td>Create a chart organized by layers</td>
<td>4/4</td>
<td>1/5 Shared back to whole group</td>
<td></td>
</tr>
<tr>
<td>Noticing that the number of layers of the haystack divides 24 or 12</td>
<td>4/4 groups</td>
<td>0/5 groups</td>
<td></td>
</tr>
<tr>
<td>Discovering that haystacks are not always “equivalent” when flipping to move height to width or length.</td>
<td>4/4 groups</td>
<td>0/5 groups</td>
<td></td>
</tr>
<tr>
<td>Discovering doubling and halving relationship between dimension () double the height and halve the length and the width)</td>
<td>2/4 groups</td>
<td>0/5 groups</td>
<td></td>
</tr>
<tr>
<td>Creates a chart organized by doubling and halving</td>
<td>¼ groups</td>
<td>0/5 groups</td>
<td></td>
</tr>
<tr>
<td>Noting that length and width add to a constant given fixed h.</td>
<td>¾ groups</td>
<td>4/5 groups</td>
<td></td>
</tr>
<tr>
<td>Noting that the length and width add to 12/h</td>
<td>2/4 groups</td>
<td>4/5 groups</td>
<td></td>
</tr>
<tr>
<td>Models the restraint for conditions for dimensions of haystack as (h+w) = 12/h [or some variation thereof]</td>
<td>¾ groups</td>
<td>4/5 groups</td>
<td></td>
</tr>
<tr>
<td>Starts with the formula above and generates values by fixing one variable and considering the possible values that work.</td>
<td></td>
<td>3/5 groups</td>
<td></td>
</tr>
<tr>
<td>Starts with formula for sum of sides of haystack 2a+2b=24, And finds all possible combinations of the as and bs.</td>
<td>1/5 group Shared back to whole group</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Prospective Teachers’ Mathematical Practices on the Chocolate Box Problem.

<table>
<thead>
<tr>
<th>Chocolate Box Problem</th>
<th>Strategies observed/Big ideas discovered</th>
<th>PETs</th>
<th>PSTs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Used the model of 3D array using layers that was set in the context</td>
<td>4/4 groups</td>
<td>2/5 groups</td>
<td></td>
</tr>
<tr>
<td>Focusing on a fixed layer and finding two numbers that multiply to 24/h</td>
<td>4/4 groups</td>
<td>2/5 groups</td>
<td></td>
</tr>
<tr>
<td>Create a chart organized by layers</td>
<td>4/4 groups</td>
<td>2/5 groups</td>
<td></td>
</tr>
<tr>
<td>Noticing that the number of layers of the chocolate box divides 24 or 12</td>
<td>4/4 groups</td>
<td>2/5 groups</td>
<td></td>
</tr>
<tr>
<td>Discovering that chocolate boxes are not always “equivalent” when flipping to move height to width or length.</td>
<td>4/4 groups</td>
<td>2/5 groups</td>
<td></td>
</tr>
<tr>
<td>Generalizes doubling and halving relationship between dimensions (double the height and halve the length or the width)</td>
<td>¾ groups shared back with whole group</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Changed the model set in the context to l(w/h) = 24 uses slices</td>
<td></td>
<td>3/5 groups</td>
<td></td>
</tr>
<tr>
<td>Fixes l and finds factors of 24/l for h and w.</td>
<td>0 groups</td>
<td>3/5 groups</td>
<td></td>
</tr>
</tbody>
</table>
Figure 1. Prospective Elementary Teachers’ Path Through Landscape

Maggies Farm (Context based strategy)
Explores hay structures using unifix cubes and paper by fixing the number of layers (height).

Maggies (Non-context based strategy)
Derives an algebraic expression that does not include dimensions by adding four areas together
\[ 2a + 2b = 24 \]

Maggie’s (Non-context based strategy)
Derives an algebraic expression that includes LWH by using one hay structure and noticing that
\[ \text{LSA} = \text{SA} – \text{top and bottom}. \]

Maggies Farm (Non-context based strategy)
Derives an algebraic expression that includes LWH by using one hay structure and noticing that
\[ \text{LSA} = 2lh + 2wh. \]

Maggie’s (Non-context based strategy)
Uses compensation to vary a and b.

Maggie’s (context based strategy)
Uses compensation to vary L and W once H is fixed.

Maggie’s (context based strategy)
Fixes dimension to vary based on its position in the algebraic expression

Classifies chocolate boxes using length and width.

Congruent rectangular prisms

\[ lw = 24 \]
\[ l \text{ is fixed and factors of } wh=24/l \text{ are found} \]

\[ (l+w)h = 12 \]
\[ (l+w) \text{ is an entity so } h \text{ gets fixed} \]

Modified halving and doubling strategy

3D array

\[ \text{LSA emerges } \]
\[ (l+w) = 12/h \]

3D array

Congruent Haystack/

1

Congruent Chocolate Boxes

2

Identifies both big ideas (congruence and covering)

4

Sees student work as right or wrong

5

Sees Miki’s intervention as raising relationship between halving and doubling and the overlap in reducing surface area.

Identifies different strategies in Miki’s second chocolate box congress but misses big ideas

Sees Miki’s second congress as a share out rather than development of ideas

Identifies some connections between big ideas developed in second chocolate box congress

\[ \text{Identifies} \]
\[ \text{both} \]
\[ \text{big} \]
\[ \text{ideas} \]

\[ \text{Sees} \]
\[ \text{Miki’s} \]
\[ \text{intervention} \]
\[ \text{as} \]
\[ \text{raising} \]
\[ \text{relationship} \]
\[ \text{between} \]
\[ \text{halving} \]
\[ \text{and} \]
\[ \text{doubling} \]
\[ \text{and the overlap in} \]
\[ \text{reducing surface area}. \]

\[ \text{Identifies} \]
\[ \text{some} \]
\[ \text{connections} \]
\[ \text{between} \]
\[ \text{big} \]
\[ \text{ideas} \]
\[ \text{developed in} \]
\[ \text{second} \]
\[ \text{chocolate} \]
\[ \text{box} \]
\[ \text{congress} \]

\[ \text{Sees} \]
\[ \text{Miki’s} \]
\[ \text{intervention} \]
\[ \text{as} \]
\[ \text{raising} \]
\[ \text{relationship} \]
\[ \text{between} \]
\[ \text{halving} \]
\[ \text{and} \]
\[ \text{doubling} \]
\[ \text{and the overlap in} \]
\[ \text{reducing surface area}. \]

\[ \text{Identifies} \]
\[ \text{some} \]
\[ \text{connections} \]
\[ \text{between} \]
\[ \text{big} \]
\[ \text{ideas} \]
\[ \text{developed in} \]
\[ \text{second} \]
\[ \text{chocolate} \]
\[ \text{box} \]
\[ \text{congress} \]

\[ \text{Sees} \]
\[ \text{Miki’s} \]
\[ \text{intervention} \]
\[ \text{as} \]
\[ \text{raising} \]
\[ \text{relationship} \]
\[ \text{between} \]
\[ \text{halving} \]
\[ \text{and} \]
\[ \text{doubling} \]
\[ \text{and the overlap in} \]
\[ \text{reducing surface area}. \]
Figure 2. Prospective Secondary Teachers’ Pathway Through Landscape
(Purple arrows indicate 4 of the 5 student groups)

Maggies Farm (Context based strategy)
Explores hay structures using unifix cubes and paper by fixing the number of layers (height)

Maggie’s (context based strategy)
Uses compensation to vary L and W once H is fixed.

Maggie’s (context based strategy)
Uses compensation to vary a and b.

Maggie’s (Non-context based strategy)
Derives an algebraic expression that includes LWH by using one hay structure and noticing that LSA = SA – top and bottom.

Maggie’s (Non-context based strategy)
Derives an algebraic expression that does not include dimensions by adding four areas together 2a+2b = 24

Maggie’s Farm (Non-context based strategy)
Derives an algebraic expression that includes LWH by using one hay structure and noticing that LSA = 2lh + 2wh.

Maggies Farm (Non-context based strategy)
Explores hay structures using unifix cubes and paper by fixing the number of layers (height)

Classifies chocolate boxes using length and width. (Non-context based strategy)

3D array

Congruent Haystacks

LSA emerges (l+w) = 12/h

Modified halving and doubling strategy

Congruent Chocolate Boxes

lw = 24
l is fixed and factors of wh=24/l are found

Congruent rectangular prisms

(Maggie’s Farm (Context based strategy)
Explores hay structures using unifix cubes and paper by fixing the number of layers (height)

Classifies chocolate boxes using layers (leads to constant perimeter within layers) (context based strategy)

Classifies chocolate boxes (context based strategy)

Identifies different strategies in Miki’s second chocolate box but misses big ideas

Sees student work as right or wrong

Sees Miki’s second congress as a share out rather than development of ideas

Identifies both big ideas (congruence and covering)

Identifies some connections between big ideas developed in second chocolate box congress

Sees Miki’s intervention as raising relationship between halving and doubling and the overlap in reducing surface area.