1. Give two different reasons why the graphs below are not isomorphic.

\[ \text{Solution:} \] Both \( A \) and \( B \) are simple graphs with six vertices and seven edges. Both graphs contain exactly two 3-cliques. Both graphs contain two vertices of degree 1, one vertex of degree 2, two vertices of degree 3, and one vertex of degree 4.

However:

(a) The vertex of degree 4 in graph \( A \) has neighbors with degree 1, 1, 3 and 3, while the vertex of degree 4 in \( B \) has neighbors with degree 1, 2, 3 and 3.

(b) In graph \( A \), every vertex of degree 3 has a vertex of degree 2 as a neighbor. In graph \( B \), there exists a vertex of degree 3 with no vertex of degree 2 as a neighbor.

(c) The graph \( B \) contains a Hamiltonian path, a walk that includes every vertex of \( B \) exactly once. Graph \( A \) contains no such path.

(d) If we remove the vertex of degree 4 from \( A \) we get a graph with three components. Removing any vertex in graph \( B \) results in a graph of no more than 2 components.

(e) The shortest distance between the leaves of \( A \) is 2. The shortest distance between the leaves of \( B \) is 3.
2. Let $G$ be a graph with exactly 10 vertices and 27 edges. Suppose that each vertex has degree 3, 5, or 7, and that there are exactly 2 vertices of degree 5. How many vertices of degree 7 does $G$ have? Justify your answer.

**Solution:**

Let $x, y$ and $z$ denote the number of vertices in $G$ of degree 3, 5 and 7, respectively. We are interested in the value of $z$.

Since all possible degrees are accounted for, we have

$$x + y + z = 10.$$  

From the statement of the problem, we have

$$y = 2.$$  

Finally, by the Hand Shaking Theorem, we have

$$3x + 5y + 7z = 2 \times 27.$$  

The three equations in three unknowns reduce to a system of two equations in two unknowns:

$$x + z = 8$$

$$3x + 7z = 44.$$  

Hence, $z = 5$. 


3. Let $G$ be a simple connected graph. The **square** of $G$, denoted $G^2$, is defined to be the graph with the same vertex set as $G$ in which two vertices $u$ and $v$ are adjacent if and only if the distance between $u$ and $v$ in $G$ is 1 or 2.

In other words, in the square of a graph $G$, vertices that were adjacent remain so. And in addition, vertices that were connected by a path of length 2 become adjacent.

(a) Show that the square of $K_{1,3}$ is $K_4$.

(b) Find two more graphs whose square is $K_4$.

**Solution:**

(a) Consider the following labeled representation of $K_{1,3}$.

```
  a   b
   ↘   ↘
   c   c
```

By definition, $K_{1,3}^2$ will keep the existing edges of $K_{1,3}$ and then join $a$ to $b$, $a$ to $c$ and $b$ to $c$.

```
  a   b
  ↘   ↘
  c   c
```

Hence $K_{1,3}^2$ is isomorphic to $K_4$.

(b) Two more examples are

```
C_4 \ (\cong K_{2,2}) \quad \rightarrow \quad \begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
```

and the graph $G$

```
\begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
```

3
4. What is the number of regular binary trees on 9 vertices?

Solution: A binary tree is an ordered rooted tree in which each vertex has at most two children. A regular binary tree is a binary tree where every vertex has an even number of children. We proved in class that the number of regular binary trees on $2n + 1$ vertices is $C_n$, the $n^{th}$ Catalan number. As $9 = 2 \times 4 + 1$, the number of regular binary trees on 9 vertices is

$$C_4 = \frac{1}{4+1} \binom{2 \times 4}{4} = \frac{1}{5} \times 70 = 14.$$
5. Find the weight of a minimum cost spanning tree for the weighted graph below.

Solution: We employ Kruskal’s algorithm. First, choose all edges weighted 1, 2, or 3, except for the one edge of weight 3 that would create a cycle. Then choose the single edge of weight 4 that does not create a cycle. Finally, add the one of the edges of weight 5 that will not create a cycle. All $12 - 1 = 11$ edges are thus accounted for and we have a minimum cost spanning tree of weight 27.
6. Determine whether the given graph is Hamiltonian. If it is, find a Hamiltonian cycle. If it is not, prove it is not.

Solution: A necessary condition for a graph to be Hamiltonian is that for each nonempty $S \subseteq V(G)$, the number of components of $G - S$ is less than or equal to the number of elements of $S$. Let $G$ be the graph given above and let $S = \{u, v\}$ be the set of vertices as indicated below.

Then the graph $G - S$ has 3 components.

Since $3 \not\leq 2$, the graph $G$ is not Hamiltonian.
7. Give a clear and careful proof of the following: A connected bipartite graph has a unique bipartition (except for, of course, interchanging the two bipartition sets).

**Solution:** Let $G$ be a connected bipartite graph. Let $X, Y, X', Y' \subseteq V(G)$, where $X, Y$ and $X', Y'$ are distinct bipartitions of the vertex set of $G$.

We then have that

(a) $X \cup Y = V(G) = X' \cup Y'$
(b) $X \cap Y = \emptyset = X' \cap Y'$
(c) $G[X], G[Y], G[X']$ and $G[Y']$ are all null graphs.

Let $A = (X \cap X') \cup (Y \cap Y')$ and $B = (Y \cap X') \cup (X \cap Y')$. Note that the set $A$ cannot be empty. If it were, it would mean that $X \subseteq Y'$ and $Y \subseteq X'$. This, in turn, implies that $X = Y'$ and $Y = X'$, i.e., the bipartition sets have been switched. Similarly, $B \neq \emptyset$.

Now let $v \in A \subseteq V(G)$. Then either $v \in X \cap X'$ or $v \in Y \cap Y'$. If $v \in X \cap X'$, then $v$ cannot be adjacent to any vertex in $Y \cap X'$, since $G[X']$ is a null graph. Furthermore, $v$ cannot be adjacent to any vertex in $X \cap Y'$, since $G[X]$ is a null graph. Hence, $v$ is not adjacent to any vertex in $B$. We reach the same conclusion when $v \in Y \cap Y'$.

So we have that no vertex in $A$ can have a vertex in $B$ as a neighbor. Since

$$A \cup B = [(X \cap X') \cup (Y \cap Y')] \cup [(Y \cap X') \cup (X \cap Y')]$$
$$= [(X \cap X') \cup (Y \cap X')] \cup [(Y \cap Y') \cup (X \cap Y')]$$
$$= [(X \cup Y) \cap X'] \cup [(Y \cup X) \cap Y']$$
$$= [V(G) \cap X'] \cup [V(G) \cap Y']$$
$$= X' \cup Y'$$
$$= V(G)$$

the graph $G$ is disconnected. Contradiction.
8. Compute the total resistance between \( u \) and \( v \) in the electrical network of resistors corresponding to the graph below, where each resistor is one ohm.

\[ \begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\end{array} \]

**Solution:** Let \( G \) be the graph corresponding to the given electrical network. We compute \( \tau(G) \) using the Matrix-Tree Theorem. First label graph \( G \) as follows.

\[ \begin{array}{ccccc}
1 & \bullet & \bullet & \bullet & 2 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
4 & \bullet & \bullet & \bullet & 3 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\end{array} \]

Then

\[ T(G) = D(G) - A(G) = \begin{pmatrix}
3 & -1 & 0 & -1 & -1 \\
-1 & 3 & -1 & 0 & -1 \\
0 & -1 & 3 & -1 & -1 \\
-1 & 0 & -1 & 3 & -1 \\
-1 & -1 & -1 & -1 & 4 \\
\end{pmatrix}. \]

By the Matrix-Tree Theorem, \( \tau(G) \) is equal to the (5, 5)-cofactor of \( T \). Thus

\[ \tau(G) = \det \begin{pmatrix}
3 & -1 & 0 & -1 \\
-1 & 3 & -1 & 0 \\
0 & -1 & 3 & -1 \\
-1 & -1 & -1 & 3 \\
\end{pmatrix}. \]

To make our lives easier, we first subtract column 2 from column 4.

\[ \tau(G) = \det \begin{pmatrix}
3 & -1 & 0 & 0 \\
-1 & 3 & -1 & -3 \\
0 & -1 & 3 & 0 \\
-1 & 0 & -1 & 3 \\
\end{pmatrix}. \]
Then we expand the determinant along the first row.

\[
\tau(G) = (-1)^{1+1} \cdot 3 \cdot \det \left( \begin{array}{ccc} 3 & -1 & -3 \\ -1 & 3 & 0 \\ 0 & -1 & 3 \end{array} \right) + (-1)^{1+2} \cdot (-1) \cdot \det \left( \begin{array}{ccc} -1 & -1 & -3 \\ 0 & 3 & 0 \\ -1 & -1 & 3 \end{array} \right)
\]

\[
= 3 \times 21 - 18 = 45
\]

Now we are interested in \(\tau_{uv}(G')\), or rather \(\tau(G'')\). That is, the number of spanning trees of the graph obtained from \(G\) by identifying the vertices \(u\) and \(v\).

We can visualize \(G''\) simply by giving \(u\) and \(v\) the same label in \(G\).

\[
\begin{array}{c}
\text{Or equivalently,}
\end{array}
\]

Then

\[
T(G'') = D(G'') - A(G'') = \begin{pmatrix} 6 & -2 & -2 & -2 \\ -2 & 3 & 0 & -1 \\ -2 & 0 & 3 & -1 \\ -2 & -1 & -1 & 4 \end{pmatrix}.
\]

Hence

\[
\tau(G'') = \det \begin{pmatrix} 6 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 0 & 3 \end{pmatrix} = 30.
\]

From the results shown in class, we now conclude that the total resistance of the network between \(u\) and \(v\) is given by.

\[
R_t = \frac{\tau(G'')}{\tau(G)} = \frac{30}{45} = \frac{2}{3}.
\]