# HOMEWORK 2

SOLUTIONS

(1) Let G be a simple graph where the vertices correspond to each of the squares of an $8 \times 8$ chess board and where two squares are adjacent if, and only if, a knight can go from one square to the other in one move. What is/are the possible degree(s) of a vertex in $G$? How many vertices have each degree? How many edges does $G$ have?

**Solution:** A knight moves two squares horizontally and one square vertically, or two squares vertically and one square horizontally, on a chessboard. By examining all possibilities, we see that; the four "corner" vertices of the graph have degree 2; the eight "edge" vertices that are next to the corners have degree 3; twenty vertices, the remaining 16 edge vertices plus four more that are next to the corners but not on the edge, have degree 4; sixteen vertices have degree 6; the remaining 16 "interior" vertices have degree 8.

The preceding paragraph answers the first two questions. For the last question we appeal to the Hand-Shaking Theorem.

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d_G(v) = \frac{1}{2}(4 \times 2 + 8 \times 3 + 20 \times 4 + 16 \times 6 + 16 \times 8) = 168$$

(2) Let $G$ be a graph with $n$ vertices and exactly $n-1$ edges. Prove that $G$ has either a vertex of degree 1 or an isolated vertex.

**Solution:** Another way to state this problem would be: Prove that $G$ has at least one vertex of degree less than 2. We will prove this by contradiction, that is we will start out by assuming that the degree of each of the vertices of $G$ is greater than or equal to 2.

Under this assumption, the Hand-Shaking Theorem gives us

$$2n - 2 = 2(n-1) = 2|E(G)| = \sum_{v \in V(G)} d_G(v) \geq \underbrace{2 + 2 + \cdots + 2}_{n \text{ times}} = 2n$$

But this is clearly impossible. Therefore our assumption must be false and there must be at least one vertex that has degree less than 2.

(3) Prove that if a graph $G$ has exactly two vertices $u$ and $v$ of odd degree, then $G$ has a $u, v$-path.

**Solution:** We will prove the statement by contradiction. That is, we first assume that $G$ is a graph with exactly two vertices of odd degree, $u$ and $v$, and that there is *no* $u, v$-path in $G$.

By definition, $G$ is a disconnected graph and the vertices $u$ and $v$ must lie in separate connected components of $G$. Thus the vertex $u$ lies in $H$, a connected subgraph of $G$, and all other vertices in $H$ (which, of course, are also vertices in $G$) have even degree. Therefore, the sum of the degrees of the vertices in $H$ is an odd number.

But by the Hand-Shaking Theorem, the sum of the degrees of the vertices in $H$ (as in any graph) must be an even number. So we have a contradiction and our assumption, that there is no $u, v$-path in $G$, must be false.

(4) Let $G$ be a simple graph. Show that either $G$ or its complement $\overline{G}$ is connected.

**Solution:** Let $G$ be a simple graph that is not connected and let $\overline{G}$ be the complement of $G$. If $u$ and $v$ are vertices in $\overline{G}$ then they are also vertices in $G$.

If $u$ and $v$ lie in separate connected components of $G$ they are not adjacent in $G$. Therefore, they must be adjacent in $\overline{G}$. Hence there exists a $u, v$-path in $\overline{G}$.

If $u$ and $v$ belong to the same connected component of $G$ and $w$ is a vertex in a different connected component of $G$, then both $u$ and $v$ are adjacent $w$ in $\overline{G}$. If $e_1$ is the edge of $\overline{G}$ with endvertices $u$ and $w$ and $e_2$ is the edge of $\overline{G}$ with endvertices $w$ and $v$, then

$$(u, e_1, w, e_2, v)$$

is a $u, v$-path in $\overline{G}$.

Since there exists a path between any two vertices of $\overline{G}$ it is a connected graph.

(5) Are any of the graphs $N_n, P_n, C_n, K_n$ and $K_{n,n}$ complements of each other?

**Solution:** Clearly, $N_n$, the simple graph with $n$ vertices and no edges, and $K_n$ the simple graph with $n$ vertices and the maximal possible edges, are complements of each other when $n > 1$.

By convention $P_1 = N_1$, so $P_1 = N_1 = \overline{N}_1 = \overline{P}_1$. Also, $P_2 = K_2$, thus $P_2$ and $N_2$ are complements of each other. The complement of $P_3$ is not connected and is clearly not the same as any of the graphs on the original list. The graph $P_4$ is isomorphic to its complement (see Problem 6).

In general, the graph $P_n$ has $n-2$ vertices of degree 2 and 2 vertices of degree 1. Therefore $\overline{P}_n$ has $n-2$ vertices of degree $n-3$ and 2 vertices of degree $n-2$. This rules out any matches for $\overline{P}_n$ when $n \geq 5$.

For $n = 1$ or 2, $C_n$ is not simple. So we don't talk about complements in those cases. Since $C_3 = K_3$, $\overline{C}_3 = N_3$. As $\overline{C}_4$ is not connected it can not be identified with any of the original graphs. The graph $C_5$ is its own complement (again see Problem 6).

We now examine $\overline{C}_n$ when $n \geq 6$. The graph $C_n$ is 2-regular. Therefore $\overline{C}_n$ is $(n-3)$-regular. Now, the graph $N_n$ is 0-regular and the graphs $P_n$ and $C_n$ are not regular at all. So no matches so far.

The only complete graph with the same number of vertices as $\overline{C}_n$ is $n-1$-regular. For $n$ even, the graph $K_{\frac{n}{2},\frac{n}{2}}$ does have the same number of vertices as $\overline{C}_n$, but it is $n$-regular. Hence, we have no matches for the complement of $C_n$ if $n \geq 6$.

We don't have to examine $K_{n,n}$ (except perhaps to identify $K_{1,1}$ with $P_2$) since if it was a complement pair with any of the previous graphs on the list we would have already seen that.

(6) Show that if a simple graph $G$ is isomorphic to its complement $\overline{G}$, then $G$ has either $4k$ or $4k+1$ vertices for some natural number $k$. Find all simple graphs on four and five vertices that are isomorphic to their complements.

**Solution:** If $G$ and $\overline{G}$ are isomorphic, they must have the same number of edges. However, the total number of edges in $G$ plus the total number of edges in $\overline{G}$ equals the number of edges in the complete graph on $n$ vertices, which is $\dfrac{n(n-1)}{2}$. Hence,

$$|E(G)| = \frac{n(n-1)}{4}.$$

This is only possible if $n$ or $n-1$ is divisible by 4. They can't both be divisible by 4 since one will be odd. So, either $n = 4k$ or $n-1 = 4k$ for some natural number $k$. In the second case, $n = 4k+1$.

From the equation above, we see that if a simple graph on four vertices is isomorphic to its complement it must have exactly three edges. This, along with trial and error, shows that the only graph with this property is the path graph $P_4$.

For the five vertex case, we make heavy use of the graph-theoretic knowledge developed in class so far and quite a bit of logical reasoning.

For a simple graph $G$ on five vertices:

(a) The degree of any vertex is either $0, 1, 2, 3$ or $4$.
(b) Since the number of edges must be $\frac{5\times 4}{4} = 5$, the sum of all vertex degrees must be 10.
(c) If $G \cong \overline{G}$, the vertex degrees (except for 2) appear in pairs. That is, for every vertex of degree 4 there must be a vertex of degree 0 and for every vertex of degree 3 there must be a vertex of degree 1. This forces the number of vertices of degree 2 to be odd. Also, we can rule out vertices of degree 4 or 0, since in a simple graph on five vertices if you have a vertex of degree 4 you cannot have a vertex of degree 0.

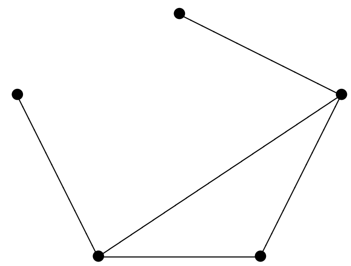
By combining the above observations, we see that set of vertex degrees for a simple graph on five vertices that is isomorphic to its complement is either

$$\{2,2,2,2,2\}, \{2,2,2,3,1\} \text{ or } \{2,3,1,3,1\}.$$

With the first set in mind, we see that $C_5$, the cycle graph on five vertices, is isomorphic to its complement.

A little more analysis allows us to rule out the second set. If a graph $G$ has exactly one vertex of degree 3 and exactly one vertex of degree 1, then the same will be true for its complement. In fact, it will be the same vertices, with their degrees switched. But in one graph the degree 3 and degree 1 vertices will be adjacent and in the complement they will not. Therefore the graphs cannot be isomorphic.

This leaves us with the $\{2, 3, 1, 3, 1\}$ case. With some experimentation, one gets the following example



Extra credit if you find a case I've missed!

(7) The complete bipartite graphs $K_{1,n}$, known as the **star graphs**, are trees. Prove that the star graphs are the only complete bipartite graphs which are trees.

**Solution:** Let $K_{m,n}$ be a complete bipartite graph such that $m, n > 1$. For $u_1, u_2, v_1, v_2 \in V(K_{m,n})$, let $u_1$ and $u_2$ be elements of the bipartition set of order $m$ and $v_1$ and $v_2$ be elements of the bipartition set of order $n$. By definition of the complete bipartite graph, there exists an edge $e_1$ with endvertices $u_1$ and $v_1$, an edge $e_2$ with endvertices $u_1$ and $v_2$, an edge $e_3$ with endvertices $u_2$ and $v_1$ and an edge $e_4$ with endvertices $u_2$ and $v_2$.

Therefore

$$c = \{u_1, e_1, v_1, e_3, u_2, e_4, v_2, e_2, u_1\}$$

is a walk in $K_{m,n}$ with distinct vertices except for the initial and final vertex, which are the same. That is, $c$ is a cycle in $K_{m,n}$. Hence, $K_{m,n}$ is not a tree.

(8) A graph $G$ is bipartite if there exists nonempty sets $X$ and $Y$ such that $V(G) = X \cup Y$, $X \cap Y = \emptyset$ and each edge in $G$ has one endvertex in $X$ and one endvertex in $Y$. Prove that any tree with at least two vertices is a bipartite graph.

**Solution:** Proof by induction. The only tree on 2 vertices is $P_2$ , which is clearly bipartite. Now assume that every tree on $n$ vertices is a bipartite graph, that is, its vertex set can be decomposed into two sets as described above.

Let $T$ be a tree on $n + 1$ vertices. From class, we know that the vertex set of $T$ contains a leaf, $v$. Furthermore, $T' = T - v$ is a tree on $n$ vertices. By the induction hypothesis, the vertex set of $T'$ can be decomposed into disjoint sets $X$ and $Y$.

Let $w$ be the neighbor of $v$ in $T$. Then $w$ is also a vertex in the graph $T'$. WLOG, assume that $w \in X \subset V(T')$. Then $X' = X$ and $Y' = Y \cup \{v\}$ is a bipartition of $V(T)$ that makes $T$ a bipartite graph.