ON TWISTED FORMS AND RELATIVE ALGEBRAIC K-THEORY

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ABSTRACT. This paper introduces a new approach to the study of certain aspects of Galois module theory by combining ideas arising from the study of the Galois structure of torsors of finite group schemes with techniques coming from relative algebraic K-theory.

1. INTRODUCTION

Let R be a Dedekind domain with field of fractions F, and suppose that $G \to \mathrm{Spec}(R)$ is a finite, flat, commutative group scheme. In recent years, there has been some interest in the study of Galois structure invariants attached to torsors of G. The original motivation for this work arose from the study of the Galois module structure of rings of integers: if $X \to \mathrm{Spec}(R)$ is a G-torsor, then we may write $X = \mathrm{Spec}(\mathfrak{C})$, and in many cases, the algebra $\mathfrak{C}$ may be viewed as being an order in the ring of integers of some (in general wildly ramified) field extension of F. The case in which G is a torsion subgroup scheme of an abelian variety is particularly interesting: here the corresponding torsors are obtained by dividing points in the Mordell-Weil group of the abelian variety, and this enables one to relate questions concerning the Galois structure of rings of integers to the arithmetic of abelian varieties. This approach was first introduced by M. J. Taylor (see [31], [30], [13]), and has since been developed in greater generality by several authors (see for example [3], [6], [12], [26]).

The main goal of this paper is to show that combining ideas arising from the study of Galois structure invariants attached to torsors of G with techniques from relative algebraic K-theory yields a natural refinement and reinterpretation of several different aspects of Galois module theory. At the same time, we shall also see that this approach also gives fresh insight into a number of old results. We remark that techniques involving relative algebraic K-theory have already played an essential role in the formulation of an equivariant version of the Tamagawa Number Conjecture of Bloch and Kato, and in the description of its consequences for Galois module theory (see [10], [11]).

We now give an outline of the contents of this paper. In §2, we recall the definition of a fibre product category, and we introduce the notion of a *generalised twisted form*. This is a generalisation of the usual notion of a twisted form or principal homogeneous space of a Hopf algebra (see Remark 2.4). An example of such an object may be given as follows. Suppose that $\mathfrak{A}$ is a finitely generated R-algebra which spans a semisimple F-algebra A, and let E be an extension of F. Then

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a generalised E-twisted $\mathfrak{A}$-form is given by a triple $[M, N; \iota]$, where M and N are finitely generated, locally free $\mathfrak{A}$-modules, and $\iota : M \otimes_R E \xrightarrow{\sim} N \otimes_R E$ is an isomorphism of $A \otimes_F E$-modules. We discuss two particular examples of generalised twisted forms that provide motivation for our later work. The first example arises via finite Galois extensions of number fields, while the second is constructed using torsors of finite, flat, commutative group schemes.

It follows from the definition that generalised twisted forms may be classified by appropriate relative algebraic K-groups. (For example, the generalised twisted forms $[M, N; \iota]$ described above yield classes in a relative algebraic K-group that we denote by $K_0(\mathfrak{A}, E)$.) In §3, we recall the basic facts concerning relative algebraic K-theory that we use, and we then give a Fröhlich-type 'Hom-description' of relative algebraic K-groups of the form $K_0(\mathfrak{A}, E)$. This description may itself be of some independent interest.

Suppose now that F is a number field. In §4, we define the notion of a metric μ on a finitely generated, locally free $\mathfrak{A}$-module M. Following [17], we then define an arithmetic classgroup $\mathrm{AC}(\mathfrak{A})$ which classifies metrised $\mathfrak{A}$-modules (M, μ). There is a natural homomorphism $\partial_{\mathfrak{A},F^c}$ from the relative algebraic K-group $K_0(\mathfrak{A}, F^c)$ to $\mathrm{AC}(\mathfrak{A})$ (where F^c denotes an algebraic closure of F). Proposition 4.11 implies that if $(M, N; \iota_1)$ is a generalised F^c-twisted $\mathfrak{A}$-form with M and N both locally free $\mathfrak{A}$-modules, then for *any* metric μ on N, the image of the class $(M, N; \iota_1)$ under $\partial_{\mathfrak{A},F^c}$ measures the difference between the elements of $\mathrm{AC}(\mathfrak{A})$ which are associated to $(M, \iota_1^*(\mu))$ and (N, μ), where $\iota_1^*(\mu)$ is the metric on M which is obtained from μ via pullback by ι_1. This result is of interest since, in many of the examples that occur in Galois module theory, canonical metrics on locally free $\mathfrak{A}$-modules can be shown to be equal to, for instance, the pullback of the trivial metric on $\mathfrak{A}$ by a natural homomorphism.

In §5, we give two applications of the approach described in this paper to the study of Galois structure of rings of integers in tamely ramified extensions. For the first application, we combine our work with results of Bley and the second-named author from [9] and we obtain a natural strengthening of a result of Chinburg, Pappas and Taylor concerning the Hermitian Galois structure of rings of integers. For the second application, we describe how certain natural torsion Galois module invariants first introduced by S. Chase (see [14]) are related to certain 'equivariant discriminants' that take values in a relative algebraic K-group.

The remainder of the paper is devoted to describing how the methods we develop may be applied to the study of the Galois structure of torsors in several different contexts. In §6, we discuss reduced resolvends associated to torsors. This notion was first introduced by L. McCulloh in a different setting (see [24]). We use a different approach from McCulloh's (see [3]), and we give a new characterisation of reduced resolvends as being primitive elements of a certain algebraic group.

We then introduce a natural refinement of the class invariant homomorphism first studied by W. Waterhouse in [33]. Our refined homomorphism takes values in a suitable relative algebraic K-group. We show that the homomorphism is injective, and that its image admits a precise functorial description in almost all cases of interest. This extends a result of the first-named author in [2].

Finally, we explain how the approach described in this paper enables one to refine McCulloh's results on realisable classes of rings of integers of tame extensions by considering invariants that lie an appropriate relative algebraic K-group, rather than in a locally free classgroup. This gives a relationship between McCulloh's realisability results, and the work of Chase concerning certain torsion Galois modules. It also yields realisability results in certain arithmetic classgroups. We find that, in general, the collection of realisable classes in the relative algebraic K-group does not form a group.

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Notation. Throughout this paper, all modules are left modules, unless explicitly stated otherwise.

For any field L, we write L^c for an algebraic closure of L, and we set $\Omega_L := \mathrm{Gal}(L^c/L)$. If L is either a number field or a local field, then O_L will denote its ring of integers.

If L is a number field, we write $S_f(L)$ (resp. $S_\infty(L)$) for the set of finite (resp. infinite) places of L. If v is any place of L, then we write L_v for the local completion of L at v. For any O_L-module P, we shall usually set $P_v := P \otimes_{O_L} O_{L_v}$.

If $S_1 \subset S_2$ are rings, and if A is any S_1-algebra, then we often write A_{S_2} for $A \otimes_{S_1} S_2$. We also often use similar notation $M_{S_2} := M \otimes_{S_1} S_2$ for any S_1-module M.

The symbol $\zeta(A)$ denotes the centre of A.

For any isomorphic A-modules M and N, we write $\mathrm{Is}_A(M, N)$ for the set of A-equivariant isomorphisms $M \xrightarrow{\sim} N$.

Throughout this paper, R denotes a Dedekind domain with field of fractions F. We write R^c for the integral closure of R in F^c. We also allow the possibility that $R = F$, in which case of course $R^c = F^c$.

Finally, we remark that if G and H are two groups, then we sometimes use the notation $g \times h$ (rather than (g, h)) for an element of the product $G \times H$. $\square$

2. Twisted Forms and Fibre Product Categories

2.1. **Fibre product categories.** Let $F_i : \mathcal{P}_i \to \mathcal{P}_3$, $i \in \{1, 2\}$ be functors between categories and consider the *fibre product category* $\mathcal{P}_4 := \mathcal{P}_1 \times_{\mathcal{P}_3} \mathcal{P}_2$. Explicitly, $\mathcal{P}_4$ is the category with objects $(L_1, L_2; \lambda)$ where L_i is an object of $\mathcal{P}_i$ for $i \in \{1, 2\}$ and $\lambda : F_1(L_1) \xrightarrow{\sim} F_2(L_2)$ is an isomorphism in $\mathcal{P}_3$. Morphisms $\alpha : (L_1, L_2; \lambda) \to (L_1', L_2'; \lambda')$ in $\mathcal{P}_4$ are pairs $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i \in \mathrm{Hom}_{\mathcal{P}_i}(L_i, L_i')$

so that the diagram

$$\begin{array}{ccc} F_1(L_1) & \xrightarrow{F_1(\alpha_1)} & F_1(L_1') \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda'} \\ F_2(L_2) & \xrightarrow{F_2(\alpha_2)} & F_2(L_2') \end{array}$$

in $\mathcal{P}_3$ commutes. Such a morphism α is an isomorphism in $\mathcal{P}_4$ if and only if the morphisms α_1 and α_2 are both isomorphisms. (The reader may consult [7, Chapter VII, §3] for more details concerning fibre product categories.)

2.2. **Generalised twisted forms.** For any ring Λ we write $\mathcal{P}_\Lambda$ for the category of finitely generated projective Λ-modules.

Recall that R is a Dedekind domain, with fraction field F. Suppose that $\mathfrak{A}$ is a finitely generated R-algebra which spans a semisimple F-algebra A. For any extension Λ of R we write $\mathcal{P}_\mathfrak{A} \times_\Lambda \mathcal{P}_\mathfrak{A}$ for the fibre product category with F_1 and F_2 both equal to the scalar extension functor $- \otimes_R \Lambda$ from $\mathcal{P}_\mathfrak{A}$ to $\mathcal{P}_{\mathfrak{A}\otimes_R\Lambda}$.

Definition 2.1. We shall refer to an object of the category $\mathcal{P}_\mathfrak{A} \times_\Lambda \mathcal{P}_\mathfrak{A}$ as a *generalised Λ-twisted $\mathfrak{A}$-form*. We abbreviate this terminology to *generalised twisted form*, or even just *twisted form*, when both $\mathfrak{A}$ and Λ are clear from the context. □

Example 2.2. (Classical) Suppose that F is a number field, and let L/F be any finite extension. We write N for the normal closure of L over F and we let $\Sigma_F(L,N)$ denote the set of distinct F-embeddings of L into N. Set $H_F^L := \prod_{\Sigma_F(L,N)} \mathbf{Z}$, and write $\mathrm{Aut}(L/F)$ for the group of F-automorphisms of L. Then there is a canonical $N[\mathrm{Aut}(L/F)]$-equivariant isomorphism

$$\pi_F^{L,N} : L \otimes_F N \xrightarrow{\sim} H_F^L \otimes_\mathbf{Z} N$$

which is given by mapping each primitive tensor $l \otimes n$ to the vector $(\sigma(l)n)_\sigma$.

Now suppose that S is any subring of F and let G be a subgroup of $\mathrm{Aut}(L/F)$. Then for any projective $S[G]$-submodule $\mathcal{L}$ of L such that $\mathcal{L} \otimes_S F = L$, the triple $(\mathcal{L}, H_F^L \otimes_\mathbf{Z} S; \pi_F^{L,N})$ is a generalised N-twisted $S[G]$-form.

In special cases there are canonical choices of $\mathcal{L}$. For example, if $S = F$, then one can take $\mathcal{L} = L$. In addition, if L is a tamely ramified Galois extension of a field K with $\mathrm{Gal}(L/K) = G$ and $S = O_F$ for any subfield F of K, then one can take $\mathcal{L}$ to be any G-stable fractional ideal of O_L (see [32]). □

Example 2.3. (Geometrical) Let R be any Dedekind domain, with field of fractions F. (We allow $R = F$.) Suppose that $G \to \mathrm{Spec}(R)$ is a finite, flat, commutative group scheme, and let G^* denote its Cartier dual. Write $\mathfrak{A} := \mathcal{O}_{G^*}$ and $A := \mathfrak{A} \otimes_R F$. It is shown by Waterhouse in [33, Sections 1 and 2] (see also [22, exposé VII] or [25]) that there is a canonical isomorphism of groups

$$H^1(\mathrm{Spec}(R), G) \simeq \mathrm{Ext}^1(G^*, \mathbf{G}_m).$$

This implies that given any G-torsor $\pi : X \to \mathrm{Spec}(R)$, we can associate to it a canonical commutative extension

$$1 \to \mathbf{G}_m \to G(\pi) \to G^* \to 1. \tag{1}$$

The scheme $G(\pi)$ is a $\mathbf{G}_m$-torsor over G^*, and we write $\mathcal{L}_\pi$ for its associated G^*-bundle.

Let $\pi_0 : G \to \mathrm{Spec}(R)$ denote the trivial G-torsor. Then, over $\mathrm{Spec}(R^c)$, the G-torsors π and π_0 become isomorphic, i.e. there is an isomorphism $X \otimes_R R^c \xrightarrow{\sim} G \otimes_R R^c$ of schemes with G-action. Hence, via the functoriality of the construction in [33, Section 2], we obtain an $\mathfrak{A} \otimes_R R^c$-equivariant isomorphism

$$\xi_\pi : \mathcal{L}_\pi \otimes_R R^c \xrightarrow{\sim} \mathfrak{A} \otimes_R R^c. \tag{2}$$

We refer to ξ_π as a *splitting isomorphism* associated to π. Then the triple $(\mathcal{L}_\pi, \mathfrak{A}; \xi_\pi)$ is a generalised R^c-twisted $\mathfrak{A}$-form. □

Remark 2.4. The line bundle $\mathcal{L}_\pi$ in Example 2.3 may be described explicitly in the following way (see [33, Section 4]). For any G-torsor $\pi : X \to \mathrm{Spec}(R)$ as above, the structure sheaf $\mathcal{O}_X$ is an $\mathcal{O}_G$-comodule, and so it is also an $\mathfrak{A}$-module (see [16, Proposition 1.3]). As an $\mathfrak{A}$-module, $\mathcal{O}_X$ is locally free of rank one, and so it gives a line bundle $\mathcal{M}_\pi$ on G^*. Then it may be shown that $\mathcal{L}_\pi = \mathcal{M}_\pi \otimes \mathcal{M}_{\pi_0}^{-1}$. The line bundle $\mathcal{M}_\pi$ is a principal homogeneous space, or twisted form of the Hopf algebra $\mathfrak{A}$ in the sense of, for example [31] or [13], and this is the motivation for Definition 2.1 above. □

3. Relative algebraic K-theory

3.1. **Basic theory.** Let Λ be any extension of R, and let $\mathfrak{A}$ be a finitely generated R-algebra which spans an F-algebra A. In this subsection, we shall recall some basic facts concerning the K-theory of categories of the form $\mathcal{P}_{\mathfrak{A}} \times_\Lambda \mathcal{P}_{\mathfrak{A}}$. Further details of these results may be found in [29, Chapter 15] and [19, Section 40B].

Definition 3.1. A short exact sequence of objects in $\mathcal{P}_{\mathfrak{A}} \times_\Lambda \mathcal{P}_{\mathfrak{A}}$ is a sequence

$$0 \to (L_1, L_2; \lambda) \xrightarrow{(\alpha_1, \alpha_2)} (L_1', L_2'; \lambda') \xrightarrow{(\alpha_1', \alpha_2')} (L_1'', L_2''; \lambda'') \to 0 \tag{3}$$

such that (α_1, α_2) and (α_1', α_2') are morphisms in $\mathcal{P}_{\mathfrak{A}} \times_\Lambda \mathcal{P}_{\mathfrak{A}}$, and such that

$$0 \to L_1 \xrightarrow{\alpha_1} L_1' \xrightarrow{\alpha_1'} L_1'' \to 0 \quad \text{and} \quad 0 \to L_2 \xrightarrow{\alpha_2} L_1' \xrightarrow{\alpha_2'} L_2'' \to 0$$

are exact sequences of R-modules. □

Definition 3.2. We write $[L_1, L_2; \lambda]$ for the isomorphism class of an object $(L_1, L_2; \lambda)$ of $\mathcal{P}_{\mathfrak{A}} \times_\Lambda \mathcal{P}_{\mathfrak{A}}$. We define the relative algebraic K-group $K_0(\mathfrak{A}, \Lambda)$ to be the abelian group with generators the isomorphism classes of objects of $\mathcal{P}_{\mathfrak{A}} \times_\Lambda \mathcal{P}_{\mathfrak{A}}$ and relations

(i) $[L_1', L_2'; \lambda'] = [L_1, L_2; \lambda] + [L_1'', L_2''; \lambda'']$ for each exact sequence (3);
(ii) $[M_1, M_2; \eta \circ \eta'] = [M_1, M_3; \eta'] + [M_3, M_2; \eta]$.

It may be shown that every element of $K_0(\mathfrak{A},\Lambda)$ is of the form $[L_1,L_2;\lambda]$ (see [29, Lemma 15.6]). However, the natural map from the set of isomorphism classes of objects of $\mathcal{P}_{\mathfrak{A}}\times_\Lambda\mathcal{P}_{\mathfrak{A}}$ to $K_0(\mathfrak{A},\Lambda)$ is very far from being injective (for example, the image of the element $[\mathfrak{A}^n,\mathfrak{A}^n;1_{\mathfrak{A}_\Lambda}]$ under this map is zero for every positive integer n). □

For any ring Λ we write $K_0T(\Lambda)$ for the Grothendieck group of the category of Λ-modules which are both finite and of finite projective dimension. Then there are natural isomorphisms

$$K_0(\mathfrak{A},F)\xrightarrow{\sim}K_0T(\mathfrak{A})\xrightarrow{\sim}\bigoplus_v K_0T(\mathfrak{A}_v) \tag{4}$$

where v runs over all finite places of R (see the discussion following (49.12) in [19]). The first isomorphism in (4) is defined in the following way. If M is any $\mathfrak{A}$-module which is both finite and of finite projective dimension, then (since R is a Dedekind domain), there exists an exact sequence of $\mathfrak{A}$-modules of the form

$$0\to O\xrightarrow{\varphi}Q\to M\to 0$$

in which P and Q are both finitely generated and projective. The first isomorphism of (4) sends (P,φ,Q) to the class of M.

For any ring S we write $K_i(S)$ $(i=0,1)$ for the algebraic K-group in degree i of the exact category $\mathcal{P}_S$. We recall that if $R\to\Lambda$ is any homomorphism of rings, then there is a long exact sequence of relative algebraic K-theory

$$K_1(\mathfrak{A})\xrightarrow{\partial^2_{\mathfrak{A},\Lambda}}K_1(\mathfrak{A}_\Lambda)\xrightarrow{\partial^1_{\mathfrak{A},\Lambda}}K_0(\mathfrak{A},\Lambda)\xrightarrow{\partial^0_{\mathfrak{A},\Lambda}}K_0(\mathfrak{A})\to K_0(\mathfrak{A}_\Lambda) \tag{5}$$

(see [29, Theorem 15.5]). Here the homomorphism $\partial^2_{\mathfrak{A},\Lambda}$ is the natural scalar extension morphism. The homomorphism $\partial^0_{\mathfrak{A},\Lambda}$ is defined by

$$\partial^0_{\mathfrak{A},\Lambda}([L_1,L_2;\lambda])=[L_1]-[L_2].$$

In order to describe $\partial^1_{\mathfrak{A},\Lambda}$, we first recall that $K_1(A_\Lambda)$ is generated by elements of the form (V,ϕ), where V is a finitely generated free $\mathfrak{A}_\Lambda$-module and $\phi\in\mathrm{Is}_{\mathfrak{A}_\Lambda}(V,V)$. If T is any projective $\mathfrak{A}$-submodule of V such that $T\otimes_R\Lambda=V$, then $\partial^1_{\mathfrak{A},\Lambda}$ is defined by setting

$$\partial^1_{\mathfrak{A},\Lambda}((V,\phi))=[T,T;\phi].$$

This definition is independent of the choice of T.

Remark 3.3. Let E be any extension of F. When A_E is a semisimple algebra, it is often convenient to compute in $K_1(A_E)$ by means of the injective 'reduced norm' map

$$\mathrm{nrd}_{A_E}:K_1(A_E)\to\zeta(A_E)^\times.$$

This map sends the element (V,ϕ) to the reduced norm of ϕ, considered as an element of the semisimple E-algebra $\mathrm{End}_{A_E}(V)$.

If $\mathfrak{A}$ is commutative, then the determinant functors over A_E and $\mathfrak{A}$ combine to induce canonical isomorphisms of $K_1(A_E)/\operatorname{im}(\partial^2_{\mathfrak{A},E})$ with $A_E^\times/\mathfrak{A}^\times$ and of $\ker[K_0(\mathfrak{A})\longrightarrow K_0(A_E)]$ with $\mathrm{Pic}(\mathfrak{A})$.

Hence the exact sequence (5) implies that in this case, the group $K_0(\mathfrak{A}, E)$ may be identified with the multiplicative group of invertible $\mathfrak{A}$-modules in A_E. $\square$

Proposition 3.4. *If E is any extension of F, then the following sequence is exact;*

$$0 \to K_1(A) \xrightarrow{f_1} K_1(A_E) \oplus K_0(\mathfrak{A}, F) \xrightarrow{f_2} K_0(\mathfrak{A}, E) \to 0. \tag{6}$$

Here $f_1(x) = (\partial^2_{A,E}(x), -\partial^1_{\mathfrak{A},F}(x))$, and $f_2(y_1, y_2) = \partial^1_{\mathfrak{A},E}(y_1) + \iota_{\mathfrak{A},E}(y_2)$, where $\iota_{\mathfrak{A},E} : K_0(\mathfrak{A}, F) \to K_0(\mathfrak{A}, E)$ is the natural scalar extension morphism.

Proof. We first observe that f_1 is injective because the scalar extension map $\partial^2_{A,E}$ is injective. Next, we note that upon comparing the exact sequences (5) with $\Lambda = E$ and $\Lambda = F$, one obtains a commutative diagram:

$$\begin{tikzcd}
K_1(\mathfrak{A}) \arrow[r, "f_3"] & K_1(A_E) \arrow[r, "\partial^1_{\mathfrak{A},E}"] & K_0(\mathfrak{A}, E) \arrow[r, "f_4"] & K_0(\mathfrak{A}) \arrow[r, "f_5"] & K_0(A_E) \\
K_1(\mathfrak{A}) \arrow[r, "f_6"] \arrow[u, equal] & K_1(A) \arrow[r, "\partial^1_{\mathfrak{A},F}"] \arrow[u, "\partial^2_{A,E}"] & K_0(\mathfrak{A}, F) \arrow[r, "f_7"] \arrow[u, "\iota_{\mathfrak{A},E}"] & K_0(\mathfrak{A}) \arrow[r] \arrow[u, equal] & K_0(A). \arrow[u, "\iota_{A,E}"]
\end{tikzcd}$$

In this diagram, the morphisms $\iota_{\mathfrak{A},E}$ and $\iota_{A,E}$ are the natural scalar extension morphisms, and are therefore injective. (We remark that in fact the injectivity of $\iota_{\mathfrak{A},E}$ follows from the injectivity of $\partial^2_{A,E}$, as may be shown by an easy diagram-chasing argument.)

We now show that f_2 is surjective. Suppose that $z \in K_0(\mathfrak{A}, E)$. Then, as $f_5(f_4(z)) = 0$ and $\iota_{A,E}$ is injective, we may choose $z_1 \in K_0(\mathfrak{A}, F)$ such that $f_7(z_1) = f_4(z)$. We then have that $z - \iota_{\mathfrak{A},E}(z_1) = \partial^1_{\mathfrak{A},E}(z_2)$ for some $z_2 \in K_1(A_E)$. It is easy to check that $f_2(z_2, z_1) = z$, and so it follows that f_2 is surjective.

It now remains to show that $\mathrm{Ker}(f_2) = \mathrm{Im}(f_1)$. Suppose that $y_1 \in K_1(A_E)$ and $y_2 \in K_0(\mathfrak{A}, F)$ are such that

$$\partial^1_{\mathfrak{A},E}(y_1) = \iota_{\mathfrak{A},E}(y_2) = y \in K_0(\mathfrak{A}, E),$$

say. We have to show that there exists $y' \in K_1(A)$ such that $\partial^2_{A,E}(y') = y_1$ and $\partial^1_{\mathfrak{A},F}(y') = y_2$.

Now $f_5(y) = f_4(\partial^1_{\mathfrak{A},E}(y_1)) = 0$, and so it follows that $f_7(y_2) = 0$. Hence there exists $y_3 \in K_1(A)$ such that $\partial^1_{\mathfrak{A},F}(y_3) = y_2$. We now deduce that $f_3(y_4) = \partial^2_{A,E}(y_3) - y_1$ for some $y_4 \in K_1(\mathfrak{A})$. Set $y' = y_3 - f_6(y_4)$. It is easy to show that $\partial^2_{A,E}(y') = y_1$ and $\partial^1_{\mathfrak{A},F}(y') = y_2$. This completes the proof. $\square$

3.2. **Hom-descriptions.** Recall that R is a Dedekind domain with field of fractions F. Suppose now that F is a number field, and let Γ be a finite group upon which Ω_F acts (possibly trivially). Let $\mathfrak{A}$ be a finitely generated R-subalgebra of $F^c[\Gamma]$ such that there is an equality

$$\mathfrak{A} \otimes_R F^c = F^c[\Gamma]. \tag{7}$$

Under this condition, for any extension E of F, we shall give a description of $K_0(\mathfrak{A}, E)$ in terms of idelic-valued functions on the ring R_Γ of F^c-valued characters of Γ. This description is modeled on the 'Hom-descriptions' of class groups introduced by Fröhlich (cf. for example [20, Chapter II]). It will be useful in later sections, and is itself perhaps also of some independent interest.

Let $\hat{\Gamma}$ denote the set of F^c-valued characters of Γ. The group Ω_F acts on R_Γ according to the following rule: if $\chi \in \hat{\Gamma}$ and $\omega \in \Omega_F$, then for each $\gamma \in \Gamma$ one has $(\omega \circ \chi)(\gamma) = \omega(\chi(\omega^{-1}(\gamma)))$.

We fix an embedding of F^c into E^c and we view each element of $\hat{\Gamma}$ as taking values in E^c. For each element a of $\mathrm{GL}_n(A_E)$ we define an element $\mathrm{Det}(a)$ of $\mathrm{Hom}(R_\Gamma, (E^c)^\times)$ in the following way: if T is a representation over F^c which has character ϕ, then

$$\mathrm{Det}(a)(\phi) := \det(T(a)).$$

This definition depends only upon the character ϕ, and not upon the choice of representation T.

We write $J_f(F^c)$ for the group of finite ideles of the field F^c, and we view $F^\times$ as being a subgroup of $J_f(F^c)$ via the natural diagonal embedding. If a is any element of $\mathrm{GL}_n(A \otimes_F J_f(E))$, then the above approach allows one to define an element $\mathrm{Det}(a)$ of $\mathrm{Hom}(R_\Gamma, J_f(F^c))$. We set

$$U_f(\mathfrak{A}) := \prod_{v \in S_f(F)} \mathfrak{A}_v^\times \subset (A \otimes_F J_f(F))^\times.$$

We then define a homomorphism

$$\Delta_{\mathfrak{A},E} : \mathrm{Det}(A^\times) \to \frac{\mathrm{Hom}(R_\Gamma, J_f(F^c))^{\Omega_F}}{\mathrm{Det}(U_f(\mathfrak{A}))} \times \mathrm{Det}(A_E^\times); \qquad \theta \mapsto (\theta, \theta^{-1}). \tag{8}$$

Theorem 3.5. *Under the above conditions, there is a natural isomorphism*

$$h_{\mathfrak{A},E} : K_0(\mathfrak{A}, E) \xrightarrow{\sim} \mathrm{Cok}(\Delta_{\mathfrak{A},E}). \tag{9}$$

Proof. From (4) and the isomorphisms of [20, Chapter II, (2.3)], it follows that there is an isomorphism

$$h_1 : K_0(\mathfrak{A}, F) \xrightarrow{\sim} \frac{\mathrm{Hom}(R_\Gamma, J_f(F^c))^{\Omega_F}}{\mathrm{Det}(U_f(\mathfrak{A}))}.$$

Also, the isomorphisms $K_1(A_\Lambda) \cong \mathrm{Det}(A_\Lambda^\times)$ of [loc. cit., Chapter II, Lemma 1.2 and Lemma 1.6] for $\Lambda \in \{E, F\}$ induce an isomorphism

$$h_2 : \frac{K_1(A_E)}{\mathrm{Im}(\partial^2_{\mathfrak{A},E})} \xrightarrow{\sim} \frac{\mathrm{Det}(A_E^\times)}{\mathrm{Det}(A^\times)}.$$

It follows that there is a natural isomorphism

$$\mathrm{Coker}(f_1) \simeq \mathrm{Coker}(\Delta_{\mathfrak{A},E}),$$

where f_1 is defined in the statement of Theorem 3.4. The desired result now follows from the fact that Theorem 3.4 implies that there is a natural isomorphism

$$\mathrm{Coker}(f_1) \xrightarrow{\sim} K_0(\mathfrak{A}, E).$$

□

Remark 3.6. An explicit description of the isomorphism $h_{\mathfrak{A},E}$ may be given as follows. Suppose then that $[X,Y;\xi]$ is an element of $K_0(\mathfrak{A},E)$ for which X_F and Y_F are free A-modules of the same rank (by [28, Lemma 15.6] any element of $K_0(\mathfrak{A},E)$ is of this shape). Then for any choice of isomorphism $\theta \in \mathrm{Is}_A(X_F, Y_F)$ one has

$$\begin{aligned}[X,Y;\xi] &= [X,Y;\theta_E] + [Y,Y;\xi\circ\theta_E^{-1}] \\ &= [X,Y;\theta_E] + \partial^1_{\mathfrak{A},E}((Y_E, \xi\circ\theta_E^{-1}))\end{aligned}$$

in $K_0(\mathfrak{A},E)$. The element $h_{\mathfrak{A},E}([X,Y;\xi])$ of $\mathrm{Cok}(\Delta_{\mathfrak{A},E})$ is then represented by the pair $(h_1([X,Y;\theta]), h_{2,E}((Y_E,\xi\circ\theta_E^{-1})))$. □

Definition 3.7. A finitely generated $\mathfrak{A}$-module M is said to be *locally free* if M_v is a free $\mathfrak{A}_v$-module for all places v in $S_f(F)$. It follows easily from this definition that if M is a locally free $\mathfrak{A}$-module, then it is projective and the associated A-module M_F is free. □

Remark 3.8. If $[X,Y;\xi]$ is an element of $K_0(\mathfrak{A},E)$ for which both X and Y are locally free $\mathfrak{A}$-modules, then one can give an explicit representative for $h_{\mathfrak{A},E}([X,Y;\xi])$ as follows.

For any ordered set of d elements $\{e^j : 1\le j\le d\}$ we write $\underline{e^j}$ for the $d\times 1$ column vector with j-th entry equal to e^j.

We choose an A-basis $\{y^j\}$ of Y_F and, for each $v\in S_f(F)$, an $\mathfrak{A}_v$-basis $\{y_v^j\}$ of Y_v and an $\mathfrak{A}_v$-basis $\{x_v^j\}$ of X_v. Let μ_v be the element of $GL_d(A_v)$ such that $\underline{y_v^j} = \mu_v\underline{y^j}$. We choose $\theta\in \mathrm{Is}_A(X_F,Y_F)$. Since $\{\theta^{-1}(y^j)\}$ is an A-basis of X_F, there exists an element λ_v of $GL_d(A_v)$ such that $\underline{x_v^j} = \lambda_v\underline{\theta^{-1}(y^j)}$. Finally we let $\mu\in GL_d(A_E)$ denote the matrix of $\xi\circ(\theta^{-1}\otimes_F E)$ with respect to the A-basis $\{y^j\}$ of Y_E. Then $h_{\mathfrak{A},E}([X,Y;\xi])$ is represented by the function

$$\left(\prod_{v\in S_f(F)} \mathrm{Det}(\lambda_v\mu_v^{-1})\right)\times \mathrm{Det}(\mu) \in \mathrm{Hom}(R_\Gamma, J_f(\mathbf{Q}^c))^{\Omega_F}\times \mathrm{Hom}(R_\Gamma,(E^c)^\times).$$

This construction will be used in the proof of Proposition 4.11. □

Examples 3.9. Suppose that F is a number field, and let $R=O_F$.

(i) Suppose that $\mathfrak{A}=R[G]$ (see Example 2.2). Then condition (7) is satisfied if we take $\Gamma=G$, viewed as a trivial Ω_F-module.

(ii) In Example 2.3, set $\Gamma = G(F^c)$, endowed with the natural Ω_F-action. Then $G\otimes_R F^c$ is a finite constant group scheme over $\mathrm{Spec}(F^c)$, and so its Cartier dual $G^*\otimes_R F^c$ is equal to $\mathrm{Spec}(F^c[\Gamma])$ (see e.g. [34, Section 2.4]). Hence $\mathfrak{A}=\mathcal{O}_{G^*}$ satisfies condition (7). Furthermore, in this case the description of Theorem 3.5 may be interpreted more explicitly as follows.

Let $J_f(A)$ denote the group of finite ideles of A, i.e. if $\mathfrak{M}$ is the (unique) maximal R-order in A, then $J_f(A)$ is the restricted direct product of the groups $A_v^\times$ with respect to the subgroups $\mathfrak{M}_v^\times$ for all places $v\in S_f(F)$. Define a map

$$\Delta'_{\mathfrak{A},E}: A^\times \to \frac{J_f(A)}{U_f(\mathfrak{A})}\times A_E^\times;\qquad a\mapsto (a)\times a^{-1}.$$

Then, taken in conjunction with Remark 3.3, the result of Theorem 3.5 implies that there is a natural isomorphism

$$K_0(\mathfrak{A}, E) \xrightarrow{\sim} \mathrm{Cok}(\Delta'_{\mathfrak{A},E}). \tag{10}$$

□

4. Metrised structures

In this section we let R denote the ring of algebraic integers in a number field F. We fix an R-order $\mathfrak{A}$ in a finite dimensional F-algebra A, and we continue to assume that the condition (7) is satisfied.

For each $\phi \in \hat{\Gamma}$ we write W_ϕ for the Wedderburn component of $F^c[\Gamma]$ which corresponds to the contragredient character $\overline{\phi}$ of ϕ. For any $F^c[\Gamma]$-module X we then set

$$X_\phi := \bigwedge_{F^c}^{\mathrm{top}}((X \otimes_{F^c} W_\phi)^\Gamma),$$

where '$\bigwedge_{F^c}^{\mathrm{top}}$' denotes the highest exterior power over F^c which is non-zero, and Γ acts diagonally on $X \otimes_{F^c} W_\phi$.

For each $v \in S_\infty(F)$ we fix an embedding $\sigma_v : F^c \to F_v^c$ (which induces the place v upon restriction to F), and we also identify F_v^c with $\mathbf{C}$.

For each complex number z we write $\overline{z}$ for its complex conjugate.

4.1. Metrised $\mathfrak{A}$-modules.

Definition 4.1. Let X be a finitely generated locally-free $\mathfrak{A}$-module. A *metric* μ on X is a set $\{\mu_{v,\phi} : v \in S_\infty(F), \phi \in \hat{\Gamma}\}$ where, for each $v \in S_\infty(F)$ and $\phi \in \hat{\Gamma}$, $\mu_{v,\phi}$ is a hermitian metric on the complex line $(X \otimes_R F^c)_\phi \otimes_{F^c,\sigma_v} \mathbf{C}$. If $x \in (X \otimes_R F^c)_\phi \otimes_{F^c,\sigma_v} \mathbf{C}$, then we shall write $\mu_{v,\phi}(x)$ for the length of x with respect to the metric $\mu_{v,\phi}$.

A *metrised* $\mathfrak{A}$*-module* is a pair (X, μ) consisting of a locally-free $\mathfrak{A}$-module X and a metric μ on X. □

Example 4.2. (Classical) In this example we adopt the notation of Example 2.2. We take $\mathfrak{A}$ to be $\mathbf{Z}[G]$, so that $\Gamma = G$ (viewed as a trivial $\Omega_{\mathbf{Q}}$-module). Via the fixed embedding $\sigma_\infty : \mathbf{Q}^c \to \mathbf{C}$ restricted to the normal closure N of L, we identify $\Sigma_{\mathbf{Q}}(L, N)$ with the set $\Sigma(L)$ of embeddings of L into $\mathbf{C}$. We remind the reader that if X is a finitely generated, locally free $\mathfrak{A}$-module, then any G-equivariant positive definite hermitian form on $X_{\mathbf{C}}$ induces a metric on X in a natural way (see e.g. [17, Definition 2.2]).

We write $\mu_{\mathbf{C}[G]}$ for the G-equivariant positive definite hermitian form on $\mathbf{C}[G]$ which satisfies

$$\mu_{\mathbf{C}[G]}(\sum_{g \in G} x_g g, \sum_{h \in G} y_h h) = \sum_{g \in G} x_g \overline{y_g}.$$

(i) There is a G-equivariant positive definite hermitian form μ_L on $H^L_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathbf{C}$ which is defined by the rule

$$\mu_L(\sum_{\sigma\in\Sigma(L)} z'_\sigma, \sum_{\tau\in\Sigma(L)} z_\tau) = \sum_{\sigma\in\Sigma(L)} z'_\sigma \overline{z_\sigma}.$$

For each $\phi \in \hat{G}$, we write $\mu_{L,\infty,\phi}$ for the metric on $(H^L_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathbf{Q}^c)_\phi$ that is obtained as the highest exterior power of the metric on

$$(H^L_{\mathbf{Q}} \otimes_{\mathbf{Z}} W_\phi)^G = (H^L_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathbf{C}) \otimes_{\mathbf{C}} (W_\phi \otimes_{\mathbf{Q}^c,\sigma_\infty} \mathbf{C})^G$$

which is induced by μ_L on $H^L_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathbf{C}$ and by the restriction of $\mu_{\mathbf{C}[G]}$ on $W_\phi \otimes_{\mathbf{Q}^c,\sigma_\infty} \mathbf{C}$.

(ii) There is a G-equivariant positive definite hermitian form h_L on $\mathbf{C} \otimes_{\mathbf{Q}} L$ which is defined by the rule

$$h_L(z_1 \otimes m, z_2 \otimes n) = z_1\overline{z_2} \sum_{\sigma\in\Sigma(L)} \sigma(m)\overline{\sigma(n)}.$$

We recall that in [17, §5.2] this form is referred to as the 'Hecke form'.

For each $\phi \in \hat{G}$ we write $h_{L,\infty,\phi}$ for the metric on $(L \otimes_{\mathbf{Q}} \mathbf{Q}^c)_\phi$ that is obtained as the highest exterior power of the metric on

$$(L \otimes_{\mathbf{Q}} W_\phi)^G \otimes_{\mathbf{Q}^c,\sigma_\infty} \mathbf{C} = ((\mathbf{C} \otimes_{\mathbf{Q}} L) \otimes_{\mathbf{C}} (W_\phi \otimes_{\mathbf{Q}^c,\sigma_\infty} \mathbf{C}))^G$$

which is induced by h_L on $\mathbf{C} \otimes_{\mathbf{Q}} L$ and by the restriction of $\mu_{\mathbf{C}[G]}$ on $W_\phi \otimes_{\mathbf{Q}^c,\sigma_\infty} \mathbf{C}$.

We set $h_{L,\bullet} := \{h_{L,\infty,\phi} : \phi \in \hat{G}\}$. If $\mathcal{L}$ is any full projective $\mathbf{Z}[G]$-sublattice of L, then the pair $(\mathcal{L}, h_{L,\bullet})$ is a metrised $\mathbf{Z}[G]$-module. □

Example 4.3. (Geometrical) In this example we adopt the notation of Example 2.3.

(i) For each $v \in S_\infty(F)$ and $\phi \in \hat{\Gamma}$, the space $(A \otimes_F F^c)_\phi \otimes_{F^c,\sigma_v} \mathbf{C}$ identifies naturally with $\mathbf{C}$ and so is endowed with a metric $\mu_{A,v,\phi}$ coming from the standard metric on $\mathbf{C}$. The set $\mu_A := \{\mu_{A,v,\phi} : v \in S_\infty(F), \phi \in \hat{\Gamma}\}$ is then a metric ('the trivial metric') on $\mathfrak{A}$.

(ii) It is shown in [6, §2] that, for each place $v \in S_\infty(F)$ and character $\phi \in \hat{\Gamma}$, the space $(\mathcal{L}_\pi \otimes_R F^c)_\phi \otimes_{F^c,\sigma_v} \mathbf{C}$ may be endowed with a canonical 'Neron metric' $||\cdot||_{v,\phi}$ which is constructed by using canonical splittings of the extension (1). (These canonical splittings are analogous to the canonical splittings of extensions of abelian varieties by tori that may be used to define Néron pairings on abelian varieties (see e.g. [23]).) Let $||\cdot||_\pi$ denote the family of metrics $\{||\cdot||_{v,\phi} : v \in S_\infty(F), \phi \in \hat{\Gamma}\}$. Then the pair $(\mathcal{L}_\pi, ||\cdot||_\pi)$ constitutes a metrised $\mathfrak{A}$-module. □

4.2. **Arithmetic class groups.** In this subsection we use idelic valued functions on R_Γ to define a group which classifies the structure of metrised $\mathfrak{A}$-modules. All of the constructions in this subsection are motivated by those of [17, §3.1, §3.2].

We write $|\Delta|_{\mathfrak{A}}$ for the homomorphism

$$\mathrm{Det}(A^\times) \to \frac{\mathrm{Hom}(R_\Gamma, J_f(F^c))^{\Omega_F}}{\mathrm{Det}(U_f(\mathfrak{A}))} \times \mathrm{Hom}\left(R_\Gamma, \prod_{v\in S_\infty(F)} \mathbf{R}^\times_{>0}\right); \qquad \theta \mapsto (\theta, |\theta|)$$

where $|\theta|$ is the homomorphism which sends each $\phi \in \hat{\Gamma}$ to the element

$$\prod_{v \in S_\infty(F)} |\sigma_v(\theta(\phi))|^{-1} \in \prod_{v \in S_\infty(F)} \mathbf{R}^\times_{>0}.$$

Remark 4.4. We caution the reader that the definition of the homomorphism $|\theta|$ given above is slightly different from that used in [17, §3.1]. Our $|\theta|(\phi)$ is equal to $|\theta|(\phi)^{-1}$ in the notation of [17]. This reflects our definitions of $\Delta_{\mathfrak{A},E}$ in (8) and $h_{\mathfrak{A},E}$ in (9). Our normalisation is also consistent with the definition of the Hermitian classgroup given in [21, Chapter II, §5]. □

Definition 4.5. We define the *arithmetic classgroup* $\mathrm{AC}(\mathfrak{A})$ of $\mathfrak{A}$ to be the cokernel of $|\Delta|_{\mathfrak{A}}$. □

Remark 4.6. When $F = \mathbf{Q}$ and $\mathfrak{A} = \mathbf{Z}[\Gamma]$ for some finite group Γ, then it is easy to show (taking into account Remark 4.4 above) that $\mathrm{AC}(\mathfrak{A})$ is isomorphic to the arithmetic classgroup $A(\mathfrak{A})$ which is described in [17, Definition 3.2]. □

We now suppose given a metrised $\mathfrak{A}$-module (M, μ). Let M have rank d over $\mathfrak{A}$, and choose an A-basis $\{m^j\}$ of M_F and, for each $v \in S_f(F)$, an $\mathfrak{A}_v$-basis $\{m^j_v\}$ of M_v. Then there exists an element λ_v of $GL_d(A_v)$ such that $\underline{m^j_v} = \lambda_v \underline{m^j}$.

For each element m of M_F we set

$$r(m) := \sum_{\gamma \in \Gamma} \gamma m \otimes \gamma \in M \otimes_R F^c[\Gamma].$$

For each element w of W_ϕ we have

$$r(m)(1 \otimes w) \in (M \otimes_R W_\phi)^\Gamma.$$

Let $\{w_{\phi,k} : k\}$ be an F^c-basis of W_ϕ which is orthonormal with respect to the restriction of $\mu_{F^c[\Gamma]}$ to W_ϕ. Then the set $\{r(m^j)(1 \otimes w_{\phi,k}) : j, k\}$ is an F^c-basis of $(M \otimes_R W_\phi)^\Gamma$, and so the wedge product

$$\bigwedge_j \bigwedge_k r(m^j)(1 \otimes w_{\phi,k})$$

is an F^c-basis of the line $(M \otimes_R F^c)_\phi$.

Definition 4.7. We define $[M, \mu]$ to be the element of $\mathrm{AC}(\mathfrak{A})$ which is represented by the homomorphism on R_Γ which sends each character $\phi \in \hat{\Gamma}$ to

$$\prod_{v \in S_f(F)} \mathrm{Det}(\lambda_v)(\phi) \times \prod_{v \in S_\infty(F)} \mu_{v,\phi}\left(\left(\bigwedge_j \bigwedge_k r(m^j)(1 \otimes w_{\phi,k})\right) \otimes_{F^c,\sigma_v} 1\right)^{\frac{1}{\phi(1)}} \in J_f(F^c) \times \prod_{v \in S_\infty(F)} \mathbf{R}^\times_{>0}.$$

It can be shown that $[M, \mu]$ is independent of the precise choices of bases $\{m^j\}$, $\{m^j_v\}$ and $\{w_{\phi,k}\}$. □

4.3. **The connection to generalised twisted forms.** Let E be any field which contains F^c and which is such that, for each $v \in S_\infty(F)$, the fixed embedding $\sigma_v : F^c \to \mathbf{C}$ factors through an embedding $\tilde{\sigma}_v : E \to \mathbf{C}$.

The map

$$\pi_E : E^\times \to \prod_{v \in S_\infty(F)} \mathbf{R}^\times_{>0}; \qquad e \mapsto \prod_{v \in S_\infty(F)} |\tilde{\sigma}_v(e)|$$

induces a homomorphism

$$\varepsilon : \mathrm{Det}(A_E^\times) \simeq \mathrm{Hom}(R_\Gamma, E^\times) \longrightarrow \mathrm{Hom}(R_\Gamma, \prod_{v \in S_\infty(F)} \mathbf{R}^\times_{>0}).$$

Then, via the Hom-descriptions of Theorem 3.5 and Definition 4.5, it is not hard to see that the map

$$\delta : \frac{\mathrm{Hom}(R_\Gamma, J_f(F^c))^{\Omega_F}}{\mathrm{Det}(U_f(\mathfrak{A}))} \times \mathrm{Det}(A_E^\times) \longrightarrow \frac{\mathrm{Hom}(R_\Gamma, J_f(F^c))^{\Omega_F}}{\mathrm{Det}(U_f(\mathfrak{A}))} \times \mathrm{Hom}(R_\Gamma, \prod_{v \in S_\infty(F)} \mathbf{R}^\times_{>0})$$

given by $x_1 \times x_2 \mapsto x_1 \times \varepsilon(x_2)$ induces a homomorphism

$$\partial_{\mathfrak{A},E} : K_0(\mathfrak{A}, E) \to \mathrm{AC}(\mathfrak{A}).$$

Lemma 4.8. *There is a natural isomorphism*

$$\ker(\partial_{\mathfrak{A},E}) \xrightarrow{\sim} \frac{\mathrm{Hom}(R_\Gamma, \ker(\pi_E))}{[\mathrm{Hom}(R_\Gamma, \ker(\pi_E)) \cap \mathrm{Det}(A^\times)] \cap \mathrm{Det}(U_f(\mathfrak{A}))}. \tag{11}$$

(In the denominator of the right-hand side of (11)*, the first intersection takes place in*

$$\mathrm{Hom}(R_\Gamma, E^\times) \simeq \mathrm{Det}(A_E^\times),$$

and then the second intersection takes place in $\mathrm{Hom}(R_\Gamma, J_f(F^c))$*.)*

Proof. Set

$$N_1 := \frac{\mathrm{Hom}(R_\Gamma, J_f(F^c))^{\Omega_F}}{\mathrm{Det}(U_f(\mathfrak{A}))} \times \mathrm{Det}(A_E^\times),$$

and write $N_2 := \delta(N_1)$. Then there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(R_\Gamma, \ker(\pi_E)) \arrow[r] & N_1 \arrow[r, "\delta"] & N_2 \arrow[r] & 0 \\
& & \mathrm{Det}(A^\times) \arrow[u, "\Delta_{\mathfrak{A},E}"] \arrow[r, equal] & \mathrm{Det}(A^\times). \arrow[u, "|\Delta|_{\mathfrak{A}}"'] &
\end{tikzcd}$$

Applying the Snake Lemma to this diagram (and taking into account Theorem 3.5 and Definition 4.5) yields the sequence

$$\mathrm{Hom}(R_\Gamma, \mathrm{Ker}(\pi_E)) \to K_0(\mathfrak{A}, E) \xrightarrow{\partial_{\mathfrak{A},E}} \mathrm{AC}(\mathfrak{A}) \to 0,$$

and it is easy to check (again using Theorem 3.5) that this implies the desired result. □

Given an explicit element of $K_0(\mathfrak{A}, E)$ we now aim to describe its image under $\partial_{\mathfrak{A},E}$.

Definition 4.9. Let X_1 and X_2 be finitely generated locally-free $\mathfrak{A}$-modules and suppose that $\xi \in \mathrm{Is}_{A_E}(X_{2,E}, X_{1,E})$. For each place $v \in S_\infty(F)$ and character $\phi \in \hat{\Gamma}$ we write

$$\xi_{v,\phi} : (X_2 \otimes_R F^c)_\phi \otimes_{F^c,\sigma_v} \mathbf{C} \xrightarrow{\sim} (X_1 \otimes_R F^c)_\phi \otimes_{F^c,\sigma_v} \mathbf{C}$$

for the isomorphism of complex lines which is induced by ξ. If h is any metric on X_1, then the *pullback* of h under ξ is defined to be the metric $\xi^*(h)$ on X_2 which satisfies

$$\xi^*(h)_{v,\phi}(z) = h_{v,\phi}(\xi_{v,\phi}(z))$$

for each $v \in S_\infty(F), \phi \in \hat{\Gamma}$ and $z \in (X_2 \otimes_R F^c)_\phi \otimes_{F^c,\sigma_v} \mathbf{C}$. □

Example 4.10. (i) (Classical) An explicit computation shows that, in the notation of Example 4.2, one has $(\pi_{\mathbf{Q}}^{L,N})^*(\mu_{L,\bullet}) = h_{L,\bullet}$.

(ii) (Geometrical) We use the notation of Example 4.3. If ξ_π is any isomorphism as in (2), then it follows from [6, Remark 2.3] that ξ_π induces an isometry between $\mathcal{L}_\pi$ endowed with the Néron metric $||\cdot||_\pi$ and $\mathfrak{A}$ endowed with the trivial metric. (Note that the map which is denoted by ξ_π in [6] is equal to $\xi_\pi^{\otimes e}$ in our present notation, where e denotes the exponent of the group scheme G.) Hence we have that $(\xi_\pi \otimes_{R^c} F^c)^*(\mu_A) = ||\cdot||_\pi$. □

Proposition 4.11. *Let $(X, Y; \xi)$ be any object of $\mathcal{P}_\mathfrak{A} \times_E \mathcal{P}_\mathfrak{A}$ for which X and Y are both locally free $\mathfrak{A}$-modules. Then, for any metric ρ on Y, one has*

$$\partial_{\mathfrak{A},E}(h_{\mathfrak{A},E}([X,Y;\xi])) = [X, \xi^*(\rho)] - [Y, \rho] \in \mathrm{AC}(\mathfrak{A}).$$

Proof. We use the notation introduced in Remark 3.8.

It is clear that $\partial_{\mathfrak{A},E}(h_{\mathfrak{A},E}([X,Y;\xi]))$ is represented by the homomorphism which sends each character $\phi \in \hat{\Gamma}$ to

$$\prod_{v \in S_f(F)} \mathrm{Det}(\lambda_v \mu_v^{-1})(\phi) \times \prod_{v \in S_\infty(F)} |\tilde{\sigma}_v(\mathrm{Det}(\mu)(\phi))|. \tag{12}$$

For each element x of $X_{i,\phi} := (X_i \otimes_R F^c)_\phi \otimes_{F^c} E$ (where $i = 1, 2$) and each place $v \in S_\infty(F)$, we write $[x]_v$ for the image of x in $X_{i,\phi} \otimes_{E,\tilde{\sigma}_v} \mathbf{C} = (X_i \otimes_R F^c)_\phi \otimes_{F^c,\sigma_v} \mathbf{C}$.

Using this notation, the element $[X, \xi^*(\rho)]$, resp. $[Y, \rho]$, is represented by the homomorphism which sends each character $\phi \in \hat{\Gamma}$ to

$$\prod_{v \in S_f(F)} \mathrm{Det}(\lambda_v)(\phi) \times \prod_{v \in S_\infty(F)} \rho_{v,\phi}\left(\left[\bigwedge_j \bigwedge_k r(\xi \circ \psi(y^j))(1 \otimes w_{\phi,k})\right]_v\right)^{\frac{1}{\phi(1)}}, \tag{13}$$

resp.

$$\prod_{v \in S_f(F)} \mathrm{Det}(\mu_v)(\phi) \times \prod_{v \in S_\infty(F)} \rho_{v,\phi}\left(\left[\bigwedge_j \bigwedge_k r(y^j)(1 \otimes w_{\phi,k})\right]_v\right)^{\frac{1}{\phi(1)}}. \tag{14}$$

For each $z = \sum_{\gamma\in\Gamma} c_\gamma\gamma \in E[\Gamma]$ we set $\overline{z} := \sum_{\gamma\in\Gamma} c_\gamma\gamma^{-1}$, and we extend this convention to matrices over $E[\Gamma]$ by applying it to individual entries. Now

$$\xi \circ \psi(y^j) = \sum_l \mu_{l,j} y^l$$

where $\mu_{l,j}$ is the (l,j)-component of the matrix μ, and so

$$r(\xi \circ \psi(y^j))(1 \otimes w_{\phi,k}) = \sum_l r(y^j)(1 \otimes \overline{\mu_{l,j}} w_{\phi,k}).$$

This implies that

$$\bigwedge_j \bigwedge_k r(\xi \circ \psi(y^j))(1 \otimes w_{\phi,k}) = \mathrm{Det}(\overline{\mu})(\overline{\phi})^{\phi(1)} \cdot \bigwedge_j \bigwedge_k r(y^j)(1 \otimes w_{\phi,k}).$$

But $\mathrm{Det}(\overline{\mu})(\overline{\phi}) = \mathrm{Det}(\mu(\phi))$, and so for each $v \in S_\infty(F)$ one has

$$\frac{\rho_{v,\phi}\left(\left[\bigwedge_j \bigwedge_k r(\xi \circ \psi(y^j))(1 \otimes w_{\phi,k})\right]_v\right)^{\frac{1}{\phi(1)}}}{\rho_{v,\phi}\left(\left[\bigwedge_j \bigwedge_k r(y^j)(1 \otimes w_{\phi,k})\right]_v\right)^{\frac{1}{\phi(1)}}} = |\tilde{\sigma}_v(\mathrm{Det}(\mu)(\phi))| \in \mathbf{R}^\times_{>0}.$$

It is now clear that the expression (13) is equal to the product of the expressions (12) and (14), and this immediately implies the claimed equality. □

5. The classical case

In this section we give two applications of our approach in the context of Example 2.2. We thus fix a finite tamely ramified Galois extensions of number fields L/K, with $F \subseteq K$, and we set $G := \mathrm{Gal}(L/K)$. Our interest is in the element

$$\delta^L_{O_F[G]}(O_L) := [O_L, H^L_F \otimes_{\mathbf{Z}} O_F; \pi^{L,N}_F] \in K_0(O_F[G], N).$$

5.1. **Arithmetic classes.** In this subsection we combine Proposition 4.11 with a result of Bley and the second named author in [9] in order to describe an explicit homomorphism which represents the element $\delta^L_{\mathbf{Z}[G]}(O_L)$. We then explain how this yields an explicit homomorphism which represents the element $[O_L, h_{L,\bullet}]$ of $\mathrm{AC}(\mathbf{Z}[G])$. We show that this description of $[O_L, h_{L,\bullet}]$ gives a natural refinement of a result of Chinburg, Pappas and Taylor in [17].

Before stating our explicit description of the class $\delta^L_{\mathbf{Z}[G]}(O_L)$ we must introduce some notation.

For each character $\phi \in \hat{G}$, we write $\tau(K,\phi)$ for the Galois-Gauss sum associated to ϕ, and we let $\epsilon(K,\phi)$ denote the epsilon constant which arises in the functional equation of the Artin L-function attached to ϕ (see [20, Chapter I, §5]). With respect to the canonical identification $\zeta(\mathbf{C}[G]) = \prod_{\hat{G}} \mathbf{C}$ (which is induced by the fixed embedding $\sigma_\infty : \mathbf{Q}^c \to \mathbf{C}$) we then set

$$\epsilon_{L/K} := (\epsilon(K,\overline{\phi}))_\phi \in \zeta(\mathbf{C}[G])^\times.$$

This element is the natural epsilon constant which is associated to the $\zeta(\mathbf{C}[G])$-valued equivariant Artin L-function of L/K, and it actually lies in $\zeta(\mathbf{R}[G])^\times$. The last fact implies that we may therefore choose an element

$$\lambda = (\lambda_\phi)_\phi \in \zeta(\mathbf{Q}[G])^\times \subset \zeta(\mathbf{C}[G])^\times$$

such that

$$\lambda\epsilon_{L/K} \in \mathrm{Im}(\mathrm{nrd}_{\mathbf{R}[G]}) \tag{15}$$

(see [9, §3.1]). We write d_K for the absolute discriminant of K.

We write $\iota : N \to \mathbf{C}$ for the embedding induced by restricting σ_∞ to N, and we use this to view $\delta^L_{\mathbf{Z}[G]}(O_L)$ as an element of $K_0(\mathbf{Z}[G], \mathbf{C})$ (using the inclusion $K_0(\mathbf{Z}[G], N) \subseteq K_0(\mathbf{Z}[G], \mathbf{C})$ induced by the functor $- \otimes_{N,\iota} \mathbf{C}$).

We can now state the main result of this section.

Theorem 5.1. *Let λ be any element of $\zeta(\mathbf{Q}[G])^\times$ which satisfies* (15)*. Then the class $\delta^L_{\mathbf{Z}[G]}(O_L)$ in $K_0(\mathbf{Z}[G], \mathbf{C})$ is represented by the homomorphism which sends each character $\phi \in \hat{G}$ to*

$$\lambda_\phi^{-1} \prod_{v \in S_f(K)} y_v(\phi) \times \lambda_\phi \tau(K, \phi)|d_K|^{\frac{\phi(1)}{2}}. \tag{16}$$

Here y_v denotes the 'unramified characteristic' function on R_G defined in [20, Chapter IV, §1].

We shall derive a description of the class $\delta^L_{\mathbf{Z}[G]}(O_L)$ from certain results in [9]. For the reader's convenience we first recall some notation from loc. cit.

Definition 5.2. (See [9, §3.2]) Set $H_L := \prod_{\Sigma(L)} \mathbf{Z}$. The groups G and $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ act on $\Sigma(L)$, and they endow H_L with the structure of $G \times \mathrm{Gal}(\mathbf{C}/\mathbf{R})$-module. For any $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-module X, write X^+ and X^- for the submodules on which complex conjugation acts by $+1$ and -1 respectively.

There is a canonical $\mathbf{C}[G \times \mathrm{Gal}(\mathbf{C}/\mathbf{R})]$-equivariant isomorphism

$$\rho_L : L \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} H_L \otimes_{\mathbf{Z}} \mathbf{C}; \qquad l \otimes z \mapsto (\sigma(l)z)_{\sigma \in \Sigma(L)}.$$

We write π_L for the composite $\mathbf{R}[G]$-equivariant isomorphism given by

$$\begin{aligned} L \otimes_{\mathbf{Q}} \mathbf{R} &= (L \otimes_{\mathbf{Q}} \mathbf{C})^+ \\ &\xrightarrow{\rho_L} (H_L \otimes_{\mathbf{Z}} \mathbf{C})^+ \\ &= (H_L^+ \otimes_{\mathbf{Z}} \mathbf{R}) \oplus (H_L^- \otimes_{\mathbf{Z}} (i \cdot \mathbf{R})) \\ &\xrightarrow{\times(1,-i)} (H_L^+ \otimes_{\mathbf{Z}} \mathbf{R}) \oplus (H_L^- \otimes_{\mathbf{Z}} \mathbf{R}) \\ &= H_L \otimes_{\mathbf{Z}} \mathbf{R}. \end{aligned}$$

We now define the *G-equivariant discriminant* $\delta_{L/K}(O_L)$ of O_L by setting

$$\delta_{L/K}(O_L) := [O_L, H_L; \pi_L] \in K_0(\mathbf{Z}[G], \mathbf{R}).$$

In what follows, we shall in fact view $\delta_{L/K}(O_L)$ as an element of $K_0(\mathbf{Z}[G],\mathbf{C})$ via the natural inclusion $K_0(\mathbf{Z}[G],\mathbf{R}) \subseteq K_0(\mathbf{Z}[G],\mathbf{C})$. □

Proof of Theorem 5.1. It follows from results in [9, see especially Lemma 7.4 and Corollary 7.7] that the class $\delta_{L/K}(O_L)$ is represented in the Hom-description of Theorem 3.5 by the element of $\mathrm{Hom}(R_G, J_f(\mathbf{Q}^c))^{\Omega_{\mathbf{Q}}} \times \mathrm{Hom}(R_G, \mathbf{C}^\times)$ which sends each character $\phi \in \hat{G}$ to

$$\lambda_\phi^{-1} \prod_{v \in S_f(K)} y_v(\phi) \times \lambda_\phi \epsilon(K,\phi),$$

where y_v is as defined in the statement of Theorem 5.1.

For each place $v \in S_\infty(K)$, we let G_v denote the decomposition group of v in G. If $\phi \in \hat{G}$, we let V_ϕ be a representation space for ϕ, and we define a complex number $w_\infty(K,\phi)$ by setting

$$w_\infty(K,\phi) := \prod_{v \in S_\infty(K)} i^{-\operatorname{codim}(V_\phi^{G_v})}.$$

We may identify H_L with $H_{\mathbf{Q}}^L$ via our fixed embedding $\iota : N \to \mathbf{C}$. By comparing the homomorphisms $\pi_{\mathbf{Q}}^{L,N} \otimes_{N,\iota} \mathbf{C}$ and $\pi_L \otimes_{\mathbf{R}} \mathbf{C}$, it may be shown that the element $\delta_{L/K}(O_L) - \delta^L_{\mathbf{Z}[G]}(O_L)$ of $K_0(\mathbf{Z}[G],\mathbf{C})$ is represented by the element of $\mathrm{Hom}(R_G, J_f(\mathbf{Q}^c))^{\Omega_{\mathbf{Q}}} \times \mathrm{Hom}(R_G, \mathbf{C}^\times)$ which sends each character $\phi \in \hat{G}$ to $1 \times w_\infty(K,\overline{\phi})$.

In addition, for each $\phi \in \hat{G}$ one has

$$\epsilon(K,\phi) = \tau(K,\phi) w_\infty(K,\overline{\phi}) |d_K|^{\frac{\phi(1)}{2}} \tag{17}$$

(see [9, (13) and the displayed formula which follows it]).

It now follows that $\delta^L_{\mathbf{Z}[G]}(O_L)$ is represented by the homomorphism which sends each character $\phi \in \hat{G}$ to

$$\lambda_\phi^{-1} \prod_{v \in S_f(K)} y_v(\phi) \times \lambda_\phi \tau(K,\phi) |d_K|^{\frac{\phi(1)}{2}},$$

as claimed. □

Corollary 5.3. *Let λ be any element of $\zeta(\mathbf{Q}[G])^\times$ which satisfies* (15). *Then the class $[O_L, h_{L,\bullet}]$ in $\mathrm{AC}(\mathbf{Z}[G])$ is represented by the homomorphism which sends each character $\phi \in \hat{G}$ to*

$$\lambda_\phi^{-1} \times |\lambda_\phi \tau(K,\phi)| (|G|^{[K:\mathbf{Q}]} |d_K|)^{\frac{\phi(1)}{2}}.$$

Proof. We first observe that Example 4.10(i) and Proposition 4.11 imply that

$$[O_L, h_{L,\bullet}] = [H_{\mathbf{Q}}^L, \mu_{L,\bullet}] + \partial_{\mathbf{Z}[G],\mathbf{C}}(\delta^L_{\mathbf{Z}[G]}(O_L)) \in \mathrm{AC}(\mathbf{Z}[G]). \tag{18}$$

It follows via an explicit computation (using, for example, [17, Lemma 2.3]) that the class $[H_{\mathbf{Q}}^L, \mu_{L,\bullet}]$ is represented by the homomorphism which sends each character $\phi \in \hat{G}$ to $1 \times |G|^{[K:\mathbf{Q}]\frac{\phi(1)}{2}}$.

Combining this with (18) and Theorem 5.1, we deduce that $[O_L, h_{L,\bullet}]$ is represented by the homomorphism which sends each $\phi \in \hat{G}$ to

$$\lambda_\phi^{-1} \prod_{v \in S_f(K)} y_v(\phi) \times |\lambda_\phi \tau(K,\phi)|(|G|^{[K:\mathbf{Q}]}|d_K|)^{\frac{\phi(1)}{2}}.$$

The desired result now follows from this description upon noting that, as a consequence of [20, Chapter IV, Theorem 29(i)], the homomorphism which sends each $\phi \in \hat{G}$ to

$$\left(\prod_{v \in S_f(K)} y_v(\phi)\right) \times 1$$

belongs to the image of $|\Delta|_{\mathfrak{A}}$. □

It is often convenient to replace $\mathrm{AC}(\mathbf{Z}[G])$ by weaker classifying groups. As an example of this, we shall now explain how Corollary 5.3 yields a natural refinement of [17, Theorem 5.9].

We first quickly recall the definition of the 'tame symplectic arithmetic classgroup' $A_T^s(\mathbf{Z}[G])$ of $\mathbf{Z}[G]$ from [loc. cit., §4.3]. We write R_G^s for the subgroup of R_G which is generated by the irreducible symplectic F^c-valued characters of G, and we write $\mathrm{Det}^s(-)$ for the restriction of the function $\mathrm{Det}(-)$ to R_G^s. Let T denote the maximal abelian tamely ramified extension of $\mathbf{Q}$ in $\mathbf{Q}^c$, and set $\hat{\mathbf{Z}} = \varprojlim_n \mathbf{Z}/n\mathbf{Z}$. Then we write $\mathrm{Det}(\hat{O}_T[G]^\times)$ for the direct limit of $\mathrm{Det}(\hat{O}_N[G]^\times)$, where N runs over all finite extensions of $\mathbf{Q}$ in T and $\hat{O}_N$ denotes the ring of integral adeles $\hat{\mathbf{Z}} \otimes O_N$.

Definition 5.4. The tame arithmetic symplectic classgroup $A_T^s(\mathbf{Z}[G])$ of $\mathbf{Z}[G]$ is defined to be the cokernel of the map

$$\mathrm{Det}^s(\mathbf{Q}[G]^\times) \longrightarrow \frac{\mathrm{Det}^s(\hat{O}_T[G]^\times)\,\mathrm{Hom}(R_G^s, J_f(\mathbf{Q}^c))^{\Omega_\mathbf{Q}}}{\mathrm{Det}^s(\hat{O}_T[G]^\times)} \times \mathrm{Hom}(R_G^s, \mathbf{R}_{>0}^\times)$$

which is induced by the diagonal map

$$\Delta^s : \mathrm{Det}^s(\mathbf{Q}[G]^\times) \longrightarrow \mathrm{Hom}(R_G^s, J_f(\mathbf{Q}^c))^{\Omega_\mathbf{Q}} \times \mathrm{Hom}(R_G^s, \mathbf{R}_{>0}^\times); \quad \theta \mapsto (\theta, \theta^{-1}).$$

□

We write

$$\theta_T^s : \mathrm{Hom}(R_G^s, J_f(\mathbf{Q}^c))^{\Omega_\mathbf{Q}} \times \mathrm{Hom}(R_G^s, \mathbf{R}_{>0}^\times) \longrightarrow A_T^s(\mathbf{Z}[G])$$

for the natural surjective map, and

$$\pi_T^s : \mathrm{AC}(\mathbf{Z}[G]) \longrightarrow A_T^s(\mathbf{Z}[G])$$

for the homomorphism induced by restricting an element of $\mathrm{Hom}(R_G, J_f(\mathbf{Q}^c))^{\Omega_\mathbf{Q}} \times \mathrm{Hom}(R_G, \mathbf{R}_{>0}^\times)$ to R_G^s, and then applying θ_T^s.

Definition 5.5. Let $\mathrm{Pf}(O_L)$ denote the 'Pfaffian' element of $\mathrm{Hom}(R_G^s, J_f(\mathbf{Q}^c))$ which is defined (in [loc. cit., Def. 5.8]) as follows: for each $\phi \in R_G^s$ one has

$$\mathrm{Pf}(O_L)(\phi)_p = \prod_{\mathfrak{p}} (-p)^{\frac{1}{2} f_{\mathfrak{p}}(\phi, \mathrm{Ind}_{I_{\mathfrak{p}}}^G u_{\mathfrak{p}})}$$

where here $\mathfrak{p}$ runs over all elements of $S_f(K)$ which divide p, $f_{\mathfrak{p}}$ is the residue class extension degree of $\mathfrak{p}$ in $K/\mathbf{Q}$, $I_{\mathfrak{p}}$ is the inertia subgroup of any fixed place of L which lies above $\mathfrak{p}$, $u_{\mathfrak{p}}$ is the augmentation character of $I_{\mathfrak{p}}$ and $(-,-)$ is the standard inner product on R_G. □

We remark that the image of $[O_L, h_{L,\bullet}]$ under π_T^s is equal to the element $\chi_T^s(O_L, \det h_\bullet)$ defined in [17, §4.3]. This implies that the following result is equivalent to [loc. cit., Th. 5.9(a)].

Corollary 5.6. *The class $\pi_T^s([O_L, h_{L,\bullet}])$ is represented by the homomorphism which sends each symplectic character $\phi \in \hat{G}$ to*

$$w_\infty(K,\phi)\mathrm{Pf}(O_L)(\phi) \times (|G|^{[K:\mathbf{Q}]}|d_K|)^{\frac{\phi(1)}{2}}.$$

Proof. Upon computing the image under π_T^s of the class represented by the explicit homomorphism described in Corollary 5.3, one finds that it suffices to prove the following result: if f denotes the homomorphism which sends each symplectic character $\phi \in \hat{G}$ to

$$\lambda_\phi w_\infty(K,\phi)\mathrm{Pf}(O_L)(\phi) \times |\lambda_\phi \tau(K,\phi)|^{-1},$$

then $\theta_T^s(f) = 0$.

To prove this we first observe that there exists a finite tamely ramified abelian extension N of $\mathbf{Q}$ and an element u of $\hat{O}_N[G]^\times$ such that, for each symplectic character $\phi \in \hat{G}$ one has

$$\mathrm{Pf}(O_L)(\phi) = \tau(K,\phi) \cdot \mathrm{Det}(u)(\phi).$$

This can be seen, for example, to be a consequence of the argument which proves [17, Proposition 5.11]. Hence one has $\theta_T^s(f) = \theta_T^s(f')$, where f' is the homomorphism which sends each symplectic character $\phi \in \hat{G}$ to

$$\lambda_\phi w_\infty(K,\phi)\tau(K,\phi) \times |\lambda_\phi \tau(K,\phi)|^{-1}.$$

Now the equality (17) combines with the containment (15) to imply that, for each symplectic character $\phi \in \hat{G}$, the real numbers λ_ϕ and $w_\infty(K,\phi)\tau(K,\phi)$ have the same sign (note that $\phi = \overline{\phi}$ if ϕ is symplectic). This implies that the element Φ of $\mathrm{Hom}(R_G^s, (\mathbf{Q}^c)^\times)^{\Omega_{\mathbf{Q}}}$ which sends each symplectic character $\phi \in \hat{G}$ to $\lambda_\phi w_\infty(K,\phi)\tau(K,\phi)$ actually belongs to $\mathrm{Det}^s(\mathbf{Q}[G]^\times)$. In addition, since for each symplectic $\phi \in \hat{G}$ one has $w_\infty(K,\phi) = \pm 1$, it also implies that $\lambda_\phi w_\infty(K,\phi)\tau(K,\phi) = |\lambda_\phi \tau(K,\phi)|$. This in turn implies that $f' = \Delta^s(\Phi)$, and hence that $\theta_T^s(f') = 0$, as required. □

5.2. **Torsion modules.** In this subsection we shall indicate how certain natural torsion Galois module invariants introduced by S. Chase (see [14], [20, Notes to Chapter III]) are related to the element $\delta^L_{O_K[G]}(O_L)$.

We first recall that there is a natural $O_L[G]$-equivariant injection

$$\psi_{L/K} : O_L \otimes_{O_K} O_L \longrightarrow \mathrm{Map}(G, O_L);\ \psi_{L/K}(l_1 \otimes l_2)(g) = g(l_1)l_2.$$

The cokernel $\mathrm{Cok}(\psi_{L/K})$ of $\psi_{L/K}$ is finite, and is of finite projective dimension as an $O_L[G]$-module because L/K is tamely ramified. Hence $\mathrm{Cok}(\psi_{L/K})$ determines a class in $K_0T(O_L[G])$. This class has been studied by Chase and other authors, and it gives information regarding the $O_K[G]$-module structure of O_L.

We write

$$i_{L/K} : K_0(O_K[G], L) \longrightarrow K_0T(O_L[G])$$

for the composite of the natural scalar extension morphism

$$K_0(O_K[G], L) \longrightarrow K_0(O_L[G], L); \quad [X, Y; i] \mapsto [X \otimes_{O_K} O_L, Y \otimes_{O_K} O_L; i]$$

and the first isomorphism of (4) with $\mathfrak{A} = O_L[G]$.

Lemma 5.7. *The class of* $\mathrm{Cok}(\psi_{L/K})$ *in* $K_0T(O_L[G])$ *is equal to the image of* $\delta^L_{O_K[G]}(O_L)$ *under* $i_{L/K}$.

Proof. There is an $O_L[G]$-equivariant isomorphism

$$\theta : \mathrm{Map}(G, O_L) \xrightarrow{\sim} H^L_K \otimes_{\mathbf{Z}} O_L; \qquad \theta(\phi) = (\phi(g))_g.$$

The stated result follows from the definition of isomorphism in $\mathcal{P}_{O_L[G]} \times_L \mathcal{P}_{O_L[G]}$ (see §2.1) and the commutativity of the following diagram in $\mathcal{P}_{L[G]}$:

$$\begin{array}{ccc} L \otimes_K L & \xrightarrow{\psi_{L/K}} & \mathrm{Map}(G, L) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \theta} \\ L \otimes_K L & \xrightarrow{\pi^{L,L}_K} & H^L_K \otimes_{\mathbf{Z}} L \end{array}$$

Here $\pi^{L,L}_K$ is as defined in Example 2.2. □

We remark that Chase has given an explicit description of $\mathrm{Cok}(\psi_{L/K})$ as an $O_L[G]$-module (see in particular Theorem 2.15 of [14]). Hence, Lemma 5.7 shows that Chase's results may be naturally reinterpreted in terms of equivariant discriminants.

6. Torsors

We shall now apply the methods developed in this paper to the 'geometrical' setting of Example 2.3.

6.1. **Reduced resolvends.** In this subsection, we shall discuss reduced resolvends coming from torsors of finite group schemes. This notion was first introduced by L. McCulloh (see [24], and the references contained therein). Our approach will be rather different from McCulloh's, and will use ideas that first appeared in [6] and [3].

We use the notation of Example 2.3. Set $\Gamma := G(R^c) = G(F^c)$.

Definition 6.1. We define

$$\begin{aligned}\mathbf{H}(\mathfrak{A}) &:= \{\alpha \in \mathfrak{A}_{R^c}^{\times} \mid \alpha^{\omega} = g_{\omega}\alpha \text{ for all } \omega \in \Omega_F, \text{ where } g_{\omega} \in \Gamma\},\\ \mathcal{H}(\mathfrak{A}) &:= \mathbf{H}(\mathfrak{A})/\Gamma,\\ H(\mathfrak{A}) &:= \mathbf{H}(\mathfrak{A})/(\Gamma \cdot \mathfrak{A}^{\times}).\end{aligned}$$

(Note that we allow the possibility that $R = F$, in which case we have $\mathfrak{A} = A$.) □

We suppose for the moment that R is a local ring, and we let $\pi : X \to \operatorname{Spec}(R)$ be a G-torsor. Then the line bundle $\mathcal{L}_{\pi}$ associated to π (see Example 2.3) is isomorphic to the trivial line bundle on G^*. Let $s_{\pi} : \mathfrak{A} \xrightarrow{\sim} \mathcal{L}_{\pi}$ be any trivialisation of $\mathcal{L}_{\pi}$. Recall from Example 2.3 that over $\operatorname{Spec}(R^c)$, π becomes isomorphic to the trivial torsor π_0, i.e. there is an isomorphism $i : X \otimes_R R^c \xrightarrow{\sim} G \otimes_R R^c$ of schemes with G-action. This isomorphism is not unique. If i' is any other such isomorphism, then $i^{-1}i' : G \otimes_R R^c \xrightarrow{\sim} G \otimes_R R^c$ is also an isomorphism of schemes with G-action, and so is given by translation by an element of Γ.

Let $\xi_{\pi} : \mathcal{L}_{\pi} \otimes_R R^c \xrightarrow{\sim} \mathfrak{A} \otimes_R R^c$ be the splitting isomorphism corresponding to i, and consider the map from $\mathfrak{A}_{R^c}$ to itself defined by

$$\mathfrak{A}_{R^c} \xrightarrow{s_{\pi} \otimes_R R^c} \mathcal{L}_{\pi} \otimes_R R^c \xrightarrow{\xi_{\pi}} \mathfrak{A}_{R^c}.$$

This is an isomorphism of $\mathfrak{A}_{R^c}$-modules, and so it is given by multiplication by some element $\mathbf{r}(s_{\pi})$ in $\mathfrak{A}_{R^c}^{\times}$. We refer to $\mathbf{r}(s_{\pi})$ as a *resolvend* of s_{π}. Note that $\mathbf{r}(s_{\pi})$ depends upon s_{π} as well as upon the choice of ξ_{π}.

If $\omega \in \Omega_F$, then $\xi_{\pi}^{\omega} = g_{\omega}\xi_{\pi}$, for some $g_{\omega} \in \Gamma$. Since $s_{\pi}^{\omega} = s_{\pi}$, we deduce that $\mathbf{r}(s_{\pi})^{\omega} = g_{\omega}\mathbf{r}(s_{\pi})$, and so $\mathbf{r}(s_{\pi}) \in \mathbf{H}(\mathfrak{A})$. As i is only well-defined up to translation by an element of Γ, it follows that changing our choice of ξ_{π} alters $\mathbf{r}(s_{\pi})$ by multiplication by an element of Γ (see also [13, Lemmas 1.2 and 1.3]). Hence the image $r(s_{\pi})$ of $\mathbf{r}(s_{\pi})$ in $\mathcal{H}(\mathfrak{A})$ depends only upon the trivialisation s_{π}. We refer to $r(s_{\pi})$ as the *reduced resolvend* of s_{π}.

We next observe that changing our choice of trivialisation s_{π} alters $\mathbf{r}(s_{\pi})$ by multiplication by an element of $\mathfrak{A}^{\times}$. It follows that the image of $r(s_{\pi})$ in the group $H(\mathfrak{A})$ depends only upon the isomorphism class of the G-torsor π.

The following result was first proved by McCulloh in the case in which G is a constant group scheme [24, §1 and §2]. In the generality given below, it is due to Byott [12, Lemma 1.11, Proposition 2.12]. A different proof of this result is given in [3].

Theorem 6.2. *Suppose that R is a local ring. Then the map*

$$H^1(\mathrm{Spec}(R), G) \to H(\mathfrak{A}); \qquad \pi \mapsto [r(s_\pi)]$$

is an isomorphism. □

We shall now show that, for any field F, the subgroup $H(A)$ of $A_{F^c}^\times/(\Gamma \cdot A^\times)$ has a natural functorial description. In order to do this, we first make the following definition.

Definition 6.3. Let R be any commutative ring, and suppose that $\mathcal{F}$ is any contravariant functor from the category of finite, flat, commutative group schemes over $\mathrm{Spec}(R)$ to the category of abelian groups.

If G is any finite, flat, commutative group scheme over R, let $m : G \times G \to G$ denote multiplication on G, and write $p_i : G \times G \to G$ $(i = 1, 2)$ for projection onto the i-th factor. Let $m^*, p_i^* : \mathcal{F}(G) \to \mathcal{F}(G \times G)$ denote the homomorphisms induced by m and p_i respectively. An element $x \in \mathcal{F}(G)$ is said to be *primitive* if $m^*(x) = p_1^*(x) \cdot p_2^*(x)$. (Note in particular that it follows that the subset of primitive elements of $\mathcal{F}(G)$ is a group.)

It is sometimes helpful to formulate this definition in terms of Hopf algebras rather than group schemes. This may be done as follows. Each functor $\mathcal{F}$ as above naturally corresponds to a covariant functor $\overline{\mathcal{F}}$ from the category of finite, flat, commutative and cocommutative Hopf algebras over R to the category of abelian groups. For any such Hopf algebra $\mathcal{A}$, let $\Delta : \mathcal{A} \to \mathcal{A} \otimes_R \mathcal{A}$ denote the comultiplication on $\mathcal{A}$. Define $i_1, i_2 : \mathcal{A} \to \mathcal{A} \otimes_R \mathcal{A}$ by $i_1(a) = a \otimes 1$ and $i_2(a) = 1 \otimes a$ for each $a \in \mathcal{A}$. Write $\Delta^*, i_1^*, i_2^* : \overline{\mathcal{F}}(\mathcal{A}) \to \overline{\mathcal{F}}(\mathcal{A} \otimes_R \mathcal{A})$ for the homomorphisms induced by Δ, i_1, and i_2 respectively. Then an element $y \in \overline{\mathcal{F}}(\mathcal{A})$ is primitive if $\Delta^*(y) = i_1^*(y) \cdot i_2^*(y)$. □

Theorem 6.4. *The group $H(A)$ is equal to the subgroup of primitive elements of the group $A_{F^c}^\times/(\Gamma \cdot A^\times)$.*

Proof. Suppose first that $a \in \mathbf{H}(A)$, with $a^\omega = g_\omega a$ for all $\omega \in \Omega_F$. We have that $A_{F^c} = F^c[\Gamma]$, and the comultiplication map $\Delta : A_{F^c} \to A_{F^c} \otimes_F A_{F^c}$ is induced by $\Delta(\gamma) = \gamma \otimes \gamma$ for $\gamma \in \Gamma$. This implies that, for all $\omega \in \Omega$,

$$\Delta(a)^\omega = (g_\omega \otimes g_\omega)\Delta(a), \quad i_1(a)^\omega = (g_\omega \otimes 1)i_1(a), \quad i_2(a)^\omega = (1 \otimes g_\omega)i_2(a).$$

Hence we have that $\Delta(a)[i_1(a)i_2(a)]^{-1} \in (A_{F^c} \otimes_F A_{F^c})^{\Omega_F} = A \otimes_F A$. This implies that the class of a in $A_{F^c}^\times/(\Gamma \cdot A^\times)$ is primitive.

Suppose conversely that $a \in A_{F^c}^\times$ represents a primitive class in $A_{F^c}^\times/(\Gamma \cdot A^\times)$. For each $\omega \in \Omega_F$, write $a^\omega = u_\omega a$. We wish to show that $u_\omega \in \Gamma$ for all ω.

Since $a \in A_{F^c}^\times$ represents a primitive class in $A_{F^c}^\times/(\Gamma \cdot A^\times)$, we have

$$\frac{\Delta(a)}{i_1(a)i_2(a)} = (g \otimes h)\beta,$$

where $g,h \in \Gamma$, and $\beta \in (A \otimes_F A)^{\times}$. Hence if $\omega \in \Omega_F$, then $\beta^{\omega} = \beta$ and

$$\left[\frac{\Delta(a)}{i_1(a)i_2(a)}\right]^{\omega} = (g^{\omega} \otimes h^{\omega})\beta = (g^{\omega}g^{-1} \otimes h^{\omega}h^{-1})\left(\frac{\Delta(a)}{i_1(a)i_2(a)}\right). \tag{19}$$

On the other hand, we also have that

$$\left[\frac{\Delta(a)}{i_1(a)i_2(a)}\right]^{\omega} = \frac{\Delta(u_{\omega})}{i_1(u_{\omega})i_2(u_{\omega})}\left(\frac{\Delta(a)}{i_1(a)i_2(a)}\right). \tag{20}$$

Hence (19) and (20) imply that

$$\frac{\Delta(u_{\omega})}{i_1(u_{\omega})i_2(u_{\omega})} = g^{\omega}g^{-1} \otimes h^{\omega}h^{-1}. \tag{21}$$

Therefore, in order to prove that $u_{\omega} \in \Gamma$, it suffices to prove the following assertion. Suppose that $u \in A_{F^c}^{\times}$ satisfies

$$\frac{\Delta(u)}{i_1(u)i_2(u)} = e \otimes f, \tag{22}$$

with $e, f \in \Gamma$. Then $u \in \Gamma$.

In order to establish this, we argue as follows. Suppose first that $u = a_{\gamma}\gamma$ for some $\gamma \in \Gamma$. Then

$$\frac{\Delta(u)}{i_1(u)i_2(u)} = \frac{a_{\gamma}(\gamma \otimes \gamma)}{a_{\gamma}^2(\gamma \otimes \gamma)} = \frac{1}{a_{\gamma}}1_{\Gamma} \otimes 1_{\Gamma},$$

and so in this case u satisfies (22) if and only if $a_{\gamma} = 1$. Hence we may assume that $u = \sum_{\gamma \in \Gamma} a_{\gamma}\gamma$ with $a_{\gamma_1}a_{\gamma_2} \neq 0$ for some $\gamma_1 \neq \gamma_2$.

Under this assumption, (22) implies that

$$\begin{aligned}\sum_{\gamma} a_{\gamma}\gamma \otimes \gamma &= (e \otimes f)\sum_{\beta,\beta'} a_{\beta}a_{\beta'}(\beta \otimes \beta') \\ &= (e \otimes f)[a_{\gamma_1}^2(\gamma_1 \otimes \gamma_1) + a_{\gamma_2}^2(\gamma_2 \otimes \gamma_2) + a_{\gamma_1}a_{\gamma_2}(\gamma_1 \otimes \gamma_2) + (\text{other terms})]. \end{aligned} \tag{23}$$

If $e = f$, then (23) implies that $a_{\gamma_1}a_{\gamma_2}(\gamma_1 \otimes \gamma_2) = 0$, whence it follows that $a_{\gamma_1}a_{\gamma_2} = 0$, which is a contradiction. On the other hand, if $e \neq f$, then we see from (23) that $a_{\gamma_1}^2 = a_{\gamma_2}^2 = 0$, which is also a contradiction. Hence $u \in \Gamma$ as claimed.

This completes the proof of the result. □

6.2. **Class invariant maps.** We retain the notation established in the previous section. In this subsection we describe a natural refinement of the class invariant homomorphism introduced by Waterhouse in [33].

Suppose that F is a number field. Then the class invariant homomorphism ψ is defined by

$$\psi : H^1(\mathrm{Spec}(R), G) \to \mathrm{Pic}(G^*); \qquad \pi \mapsto (\mathcal{L}_{\pi}).$$

Remark 6.5. In general ψ is not injective (see e.g. [1], [26], [28]). It is known that the image of ψ is always contained in the subgroup $\mathrm{PPic}(G^*)$ of primitive classes of $\mathrm{Pic}(G^*)$ (see e.g. [15]). It is shown by the first-named author in [2] that if the composition series of every geometric fibre of $G \to \mathrm{Spec}(R)$ of residue characteristic 2 does not contain a factor of local-local type, then the image of ψ is in fact equal to $\mathrm{PPic}(G^*)$. □

Now suppose that $\pi : X \to \mathrm{Spec}(R)$ is a G-torsor, and fix a choice of isomorphism ξ_π as in (2). For ease of notation, we use the same symbol ξ_π for the map $\mathcal{L}_\pi \otimes_R F^c \xrightarrow{\sim} A_{F^c}$ induced by (2). In general the element $[\mathcal{L}_\pi, \mathfrak{A}; \xi_\pi] \in K_0(\mathfrak{A}, R^c)$ depends upon the choice of ξ_π.

Recall that there is a natural morphism

$$\iota_{\mathfrak{A},R^c} : K_0(\mathfrak{A}, R^c) \to K_0(\mathfrak{A}, F^c)$$

arising via extension of scalars from R to F. The image of $\iota_{\mathfrak{A},R^c}([\mathcal{L}_\pi, \mathfrak{A}; \xi_\pi])$ under the isomorphism (10) with $E = F^c$ may be described as follows. Choose any trivialisation $s_\pi : A \xrightarrow{\sim} \mathcal{L}_\pi \otimes_R F$. For each finite place v of F, choose a trivialisation $t_{\pi,v} : \mathfrak{A}_v \xrightarrow{\sim} \mathcal{L}_\pi \otimes_R R_v$. It is easy to check that the isomorphism

$$A_v \xrightarrow{t_{\pi,v}} \mathcal{L}_\pi \otimes_R F_v \xrightarrow{(s_\pi \otimes_F F_v)^{-1}} A_v$$

is given by multiplication by $\mathbf{r}(t_{\pi,v}).\mathbf{r}(s_\pi)^{-1} \in A_v^\times$. Then Remark 3.8 implies that a representative of the image of $\iota_{\mathfrak{A},R^c}([\mathcal{L}_\pi, \mathfrak{A}; \xi_\pi])$ under the isomorphism (10) is given by

$$\prod_{v \in S_f(F)} (\mathbf{r}(t_{\pi,v}).\mathbf{r}(s_\pi)^{-1}) \times \mathbf{r}(s_\pi) \in J_f(A) \times A_{F^c}^\times. \tag{24}$$

Observe that if we change the choice of splitting isomorphism ξ_π, the element $\mathbf{r}(s_\pi)$ is altered by multiplication by an element of Γ. This motivates the following definition:

Definition 6.6. Let $\Delta''_{\mathfrak{A},F^c}$ denote the composition of the homomorphism

$$\Delta'_{\mathfrak{A},F^c} : A^\times \longrightarrow \frac{J_f(A)}{U_f(\mathfrak{A})} \times A_{F^c}^\times$$

defined in Example 3.9(ii) with the quotient map

$$\frac{J_f(A)}{U_f(A)} \times A_{F^c}^\times \longrightarrow \frac{J_f(A)}{U_f(A)} \times \frac{A_{F^c}^\times}{\Gamma}.$$

We define $\overline{K}_0(\mathfrak{A}, F^c)$ to be the quotient of $K_0(\mathfrak{A}, F^c)$ defined by the isomorphism

$$\overline{K}_0(\mathfrak{A}, F^c) \simeq \frac{\dfrac{J_f(A)}{U_f(\mathfrak{A})} \times \dfrac{A_{F^c}^\times}{\Gamma}}{\mathrm{Im}(\Delta''_{\mathfrak{A},F^c})}. \tag{25}$$

We write $\overline{K}_0(\mathfrak{A}, R^c)$ for the image of $K_0(\mathfrak{A}, R^c)$ under the composite of the scalar extension map $\iota_{\mathfrak{A},R^c}$ and the natural projection map $K_0(\mathfrak{A}, F^c) \to \overline{K}_0(\mathfrak{A}, F^c)$. □

Lemma 6.7. *The natural projection map $K_0(\mathfrak{A}, F^c) \to \overline{K}_0(\mathfrak{A}, F^c)$ is bijective if and only if the points of $\Gamma = G(F^c)$ are fixed by Ω_F.*

Proof. It is clear that the stated projection is bijective if and only if $\Gamma \subseteq \mathfrak{A}_v^\times$ for all $v \in S_f(F)$, i.e. if and only if $\Gamma \subseteq \mathfrak{A}^\times$. But this last condition obtains if and only if the points of Γ are fixed by Ω_F. □

It follows from (24) that the class of $\iota_{\mathfrak{A},R^c}([\mathcal{L}_\pi,\mathfrak{A};\xi_\pi])$ in $\overline{K}_0(\mathfrak{A},F^c)$ is represented by

$$\prod_{v\in S_f(F)} (\mathbf{r}(t_{\pi,v}).\mathbf{r}(s_\pi)^{-1}) \times \mathbf{r}(s_\pi) \in J_f(A)\times\mathcal{H}(A).$$

It is easy to check that this class is independent of the choice of splitting isomorphism ξ_π and that it depends only upon the isomorphism class of the G-torsor $\pi: X\to \mathrm{Spec}(R)$.

Theorem 6.8. *The map*

$$\boldsymbol{\psi}: H^1(\mathrm{Spec}(R),G)\to \overline{K}_0(\mathfrak{A},R^c); \qquad \pi\mapsto \iota_{\mathfrak{A},R^c}([\mathcal{L}_\pi,\mathfrak{A};\xi_\pi])$$

is an injective group homomorphism.

Proof. Let us first show that $\boldsymbol{\psi}$ is a group homomorphism. Suppose that π and π' are G-torsors. Let ξ_π and $\xi_{\pi'}$ denote splitting isomorphisms associated to π and π' respectively. It follows from the functoriality of Waterhouse's construction in [33] that there is a canonical isomorphism $\mathcal{L}_\pi\otimes_{\mathfrak{A}}\mathcal{L}_{\pi'}\simeq \mathcal{L}_{\pi\cdot\pi'}$ and that the map

$$\xi_\pi\otimes_{\mathfrak{A}_{R^c}}\xi_{\pi'}: (\mathcal{L}_\pi\otimes_R R^c)\otimes_{\mathfrak{A}_{R^c}}(\mathcal{L}_{\pi'}\otimes_R R^c)\xrightarrow{\sim}\mathfrak{A}_{R^c}\otimes_{\mathfrak{A}_{R^c}}\mathfrak{A}_{R^c}\xrightarrow{\sim}\mathfrak{A}_{R^c}$$

is a splitting isomorphism associated to $\pi\cdot\pi'$. Since in $K_0(\mathfrak{A},R^c)$, we have

$$[\mathcal{L}_\pi,\mathfrak{A};\xi_\pi]+[\mathcal{L}_{\pi'},\mathfrak{A};\xi_{\pi'}]=[\mathcal{L}_\pi\otimes_{\mathfrak{A}}\mathcal{L}_{\pi'},\mathfrak{A};\xi_\pi\otimes_{\mathfrak{A}_{R^c}}\xi_{\pi'}],$$

(cf. for example [8, proof of Lemma 2.6(i)]), it follows that

$$\boldsymbol{\psi}(\pi)+\boldsymbol{\psi}(\pi')=\boldsymbol{\psi}(\pi\cdot\pi'),$$

as required.

To show that $\boldsymbol{\psi}$ is injective, we argue as follows. If we compose the natural injection $H^1(\mathrm{Spec}(R),G)\to H^1(F,G)$ with the isomorphism $H^1(F,G)\to H(A)$ afforded by Theorem 6.2, then we obtain an injection $H^1(\mathrm{Spec}(R),G)\to H(A)$. It follows easily from the definitions that this injection is the same as the homomorphism obtained by composing $\boldsymbol{\psi}$ with the inclusion $\overline{K}_0(\mathfrak{A},R^c)\subseteq\overline{K}_0(\mathfrak{A},F^c)$ and the natural projection

$$\overline{K}_0(\mathfrak{A},F^c)\to\frac{A^\times_{F^c}}{\Gamma\cdot A^\times}.$$

This implies that $\boldsymbol{\psi}$ is injective. □

Remark 6.9. Let $\phi\in\hat{\Gamma}$. Then for each $\gamma\in\Gamma$ and $v\in S_\infty(F)$, the complex number $\sigma_v(\phi(\gamma))$ is a root of unity and hence has absolute value 1. This implies that the morphism

$$\partial_{\mathfrak{A},F^c}: K_0(\mathfrak{A},F^c)\to \mathrm{AC}(\mathfrak{A})$$

defined in §4.3 induces a morphism $\overline{K}_0(\mathfrak{A},F^c)\to\mathrm{AC}(\mathfrak{A})$. Upon composing this morphism with $\boldsymbol{\psi}$ we obtain a morphism

$$\hat{\psi}: H^1(\mathrm{Spec}(R),G)\to\mathrm{AC}(\mathfrak{A})\simeq\widehat{\mathrm{Pic}}(G^*)$$

which was first introduced and studied by the first-named author and G. Pappas in [6]. In turn, upon composing $\hat{\psi}$ with the natural surjection $\mathrm{AC}(\mathfrak{A}) \to \mathrm{Pic}(G^*)$ we obtain the homomorphism

$$\psi : H^1(\mathrm{Spec}(R), G) \to \mathrm{Pic}(G^*)$$

which was introduced by Waterhouse in [33]. □

Write $P\overline{K}_0(\mathfrak{A}, R^c)$ for the subgroup of primitive elements of $\overline{K}_0(\mathfrak{A}, R^c)$.

Theorem 6.10. *(i) The image of $\boldsymbol{\psi}$ lies in $P\overline{K}_0(\mathfrak{A}, R^c)$.*

(ii) Suppose that the composition series of every geometric fibre of $G \to \mathrm{Spec}(R)$ of residue characteristic 2 does not contain a factor of local-local type. Then the image of $\boldsymbol{\psi}$ is equal to $P\overline{K}_0(\mathfrak{A}, R^c)$.

Proof. (i) Recall from Example 2.3 that we have a canonical isomorphism

$$\mathrm{Ext}^1(G^*, \mathbf{G}_m) \simeq H^1(\mathrm{Spec}(R), G).$$

Composing this isomorphism with $\boldsymbol{\psi}$ gives a homomorphism

$$\varkappa_{G^*} : \mathrm{Ext}^1(G^*, \mathbf{G}_m) \to \overline{K}_0(\mathfrak{A}, R^c)$$

whose image is the same as that of $\boldsymbol{\psi}$. It follows from the functoriality of Waterhouse's construction in [33] that the homomorphism $\varkappa_{G^*}$ is functorial in G^*.

Let $m : G^* \times G^* \to G^*$ denote multiplication on G^*, and write $p_i : G^* \times G^* \to G^*$ $(i = 1, 2)$ for projection onto the i-th factor. Let

$$m^*, p_i^* : \mathrm{Ext}^1(G^*, \mathbf{G}_m) \to \mathrm{Ext}^1(G^* \times G^*, \mathbf{G}_m)$$

be the homomorphisms induced by m and p_i. Then it follows from standard properties of the functor Ext^1 that $m^*(x) = p_1^*(x) + p_2^*(x)$ for all $x \in \mathrm{Ext}^1(G^*, \mathbf{G}_m)$ (see e.g. [27, Proposition 1 on p. 163, and p. 182]). Since $\varkappa_{G^*}$ is functorial in G^*, this implies that $\varkappa_{G^*}(x)$ is a primitive element of $\overline{K}_0(\mathfrak{A}, R^c)$.

(ii) It is shown in [3] that there is a natural isomorphism $\mathrm{Ker}(\psi) \simeq H(\mathfrak{A})$. From [2, Theorem 1.3], we know that, under the given hypotheses, we have that $\mathrm{Im}(\psi) = \mathrm{PPic}(G^*)$. Since $\boldsymbol{\psi}$ is injective, this implies that there is an exact sequence

$$0 \to H(\mathfrak{A}) \to \mathrm{Im}(\boldsymbol{\psi}) \to \mathrm{PPic}(G^*) \to 0. \tag{26}$$

Next, we observe that upon comparing the exact sequences (5) with $\Lambda = R^c$ and $\Lambda = F^c$, we obtain a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{cok}[K_1(\mathfrak{A}) \to K_1(\mathfrak{A}_{R^c})] \arrow[r] \arrow[d] & K_0(\mathfrak{A}, R^c) \arrow[r] \arrow[d, "\iota_{\mathfrak{A}, R^c}"'] & \ker[K_0(\mathfrak{A}) \to K_0(\mathfrak{A}_{R^c})] \arrow[r] \arrow[d, "\subseteq"'] & 0 \\
0 \arrow[r] & \mathrm{cok}[K_1(\mathfrak{A}) \to K_1(A_{F^c})] \arrow[r] & K_0(\mathfrak{A}, F^c) \arrow[r] & \ker[K_0(\mathfrak{A}) \to K_0(A_{F^c})] \arrow[r] & 0.
\end{tikzcd}$$

Now, since $\mathfrak{A}$ is commutative, the second and fourth terms of the lower sequence can be identified with $A_{F^c}^{\times}/\mathfrak{A}^{\times}$ and $\mathrm{Pic}(\mathfrak{A})$ respectively (cf. the end of Remark 3.3). Further, the image of the natural composite morphism

$$K_1(\mathfrak{A}_{R^c}) \to K_1(A_{F^c}) \to \mathrm{cok}[K_1(\mathfrak{A}) \to K_1(A_{F^c})] \cong A_{F^c}^{\times}/\mathfrak{A}^{\times}$$

is equal to $\mathfrak{A}_{R^c}^{\times}/\mathfrak{A}^{\times}$. Upon identifying $\mathrm{Pic}(\mathfrak{A})$ with $\mathrm{Pic}(G^*)$ the above diagram implies that there is an exact sequence

$$1 \to \frac{\mathfrak{A}_{R^c}^{\times}}{\mathfrak{A}^{\times}} \to \iota_{\mathfrak{A},R^c}(K_0(\mathfrak{A}, R^c)) \to \mathrm{Pic}(G^*).$$

It is easy to see that this in turn induces an exact sequence

$$1 \to \frac{\mathfrak{A}_{R^c}^{\times}}{\Gamma \cdot \mathfrak{A}^{\times}} \to \overline{K}_0(\mathfrak{A}, R^c) \to \mathrm{Pic}(G^*). \tag{27}$$

Since $\mathrm{Im}(\boldsymbol{\psi}) \subseteq P\overline{K}_0(\mathfrak{A}, R^c)$ it follows from (26) that the natural map $P\overline{K}_0(\mathfrak{A}, R^c) \to \mathrm{PPic}(G^*)$ is surjective. Hence Theorem 6.4 implies that (27) induces an exact sequence

$$0 \to H(\mathfrak{A}) \to P\overline{K}_0(\mathfrak{A}, R^c) \to \mathrm{PPic}(G^*) \to 0. \tag{28}$$

We now see from (26) and (28) that $\mathrm{Im}(\boldsymbol{\psi}) = P\overline{K}_0(\mathfrak{A}, R^c)$, as asserted. □

6.3. **Rings of integers.** In this subsection we shall briefly explain how McCulloh's results on realisable classes of tame extensions in [24] can be naturally lifted to the corresponding relative algebraic K-group. Proposition 5.7 implies that such realisability results in the relative algebraic K-group are related to those of Chase in [14].

Suppose that F is a number field with ring of integers R. Let G be any finite, constant abelian group scheme over $\mathrm{Spec}(R)$, and set $\Gamma := G(F^c)$. Then in this case, we have $G = \mathrm{Spec}(\mathfrak{B})$ and $G^* = \mathrm{Spec}(\mathfrak{A})$, where

$$\mathfrak{B} = \mathrm{Map}(\Gamma, R), \qquad \mathfrak{A} = R[\Gamma].$$

Write G_F and G_F^* for the generic fibres of G and G^* respectively. Note that, since G_F is a constant group scheme over $\mathrm{Spec}(F)$, Lemma 6.7 implies that $\overline{K}_0(\mathfrak{A}, F^c)$ is canonically isomorphic to $K_0(\mathfrak{A}, F^c)$.

Now suppose that $\pi : X \to \mathrm{Spec}(F)$ is a G_F-torsor, and write L_π for its associated line bundle on G_F^*. Then $X = \mathrm{Spec}(C_\pi)$, where C_π is Galois algebra extension of F with Galois group Γ.

Definition 6.11. We say that π is *tame* if it is trivialised by a tamely ramified extension of F. Equivalently, π is tame if and only if the extension C_π/F is tamely ramified.

We write $H_t^1(\mathrm{Spec}(F), G_F)$ for the subgroup of $H^1(\mathrm{Spec}(F), G_F)$ consisting of isomorphism classes of tame G_F-torsors. □

Let O_π denote the integral closure of R in C_π. If π is tame, then Noether's theorem (see e.g. [20, Chapter I, §3]) implies that O_π is a locally free, rank one $\mathfrak{A}$-module, and so it yields a line bundle on G^* which we shall denote by $\mathcal{M}_\pi$. Let $\pi_0 : G_F \to \mathrm{Spec}(F)$ denote the trivial G_F-torsor, and set

$$\overline{\mathcal{L}}_\pi := \mathcal{M}_\pi \otimes \mathcal{M}_{\pi_0}^{-1}.$$

It follows from the definitions that there is a natural isomorphism $\overline{\mathcal{L}}_\pi \otimes_R F \simeq L_\pi$. If we make this identification, then any choice of splitting isomorphism ξ_π for π gives a class $[\overline{\mathcal{L}}_\pi, \mathfrak{A}; \xi_\pi]$ in $K_0(\mathfrak{A}, F^c)$. This class is independent of the choice of ξ_π. Hence we obtain a map

$$\boldsymbol{\psi}_t : H_t^1(\mathrm{Spec}(F), G_F) \to K_0(\mathfrak{A}, F^c); \qquad \pi \mapsto [\overline{\mathcal{L}}_\pi, \mathfrak{A}; \xi_\pi]. \tag{29}$$

If F is a number field, then a representative in $J_f(A) \times A_{F^c}^\times$ of $\boldsymbol{\psi}_t(\pi)$ may be given exactly as in the previous section. It may be shown via an argument very similar to that given in Theorem 6.8 that the map $\boldsymbol{\psi}_t$ is injective. However, in general, $\boldsymbol{\psi}_t$ is not a group homomorphism (see Remark 6.13 below).

Definition 6.12. We say that an element of $K_0(\mathfrak{A}, F^c)$ is *realisable* if it lies in the image of $\boldsymbol{\psi}_t$. □

The general strategy for describing the set of realisable classes in $K_0(\mathfrak{A}, F^c)$ is as follows. Let

$$\mathbf{j} : J_f(A) \times [A_{F^c}^\times/\Gamma] \to K_0(\mathfrak{A}, F^c)$$

denote the obvious quotient map afforded by (25) (together with Lemma 6.7), and write $\mathcal{H}(\mathbb{A}(A))$ for the restricted direct product of the groups $\mathcal{H}(A_v)$ with respect to the subgroups $\mathcal{H}(\mathfrak{A}_v)$. Then the obvious natural maps $A_v^\times \to \mathcal{H}(A_v)$ induce a homomorphism

$$\mathrm{rag} : J_f(A) \to \mathcal{H}(\mathbb{A}(A)).$$

Suppose that $\underline{c} := c_{\mathrm{fin}} \times c_\infty \in J_f(A) \times [A_{F^c}^\times/\Gamma]$ satisfies $\mathbf{j}(\underline{c}) \in \mathrm{Im}(\boldsymbol{\psi}_t)$. McCulloh shows that the idele $\mathrm{rag}(c_{\mathrm{fin}})$ admits a special type of decomposition (see [24, Theorem 5.6]). It is a straightforward exercise to use the results of [24] to characterise the image of $\boldsymbol{\psi}_t$ in a manner similar to that given in [24, Theorem 6.7]. We omit the details in order to keep this paper to a reasonable length.

Remarks 6.13. (i) McCulloh has shown that the image of the map obtained by composing $\boldsymbol{\psi}_t$ with the quotient map $K_0(\mathfrak{A}, F^c) \to \mathrm{Pic}(G^*)$ is a subgroup of $\mathrm{Pic}(G^*)$ (see [24, Corollary 6.20]). On the other hand, by using the description of the image of $\boldsymbol{\psi}_t$ mentioned above, it may be shown that in general, $\mathrm{Im}(\boldsymbol{\psi}_t)$ is not a group. This situation is very similar to that which obtains in the case of realisable classes in classgroups of sheaves discussed in [4, Remark 2.10(iii)].

(ii) By composing $\boldsymbol{\psi}_t$ with the natural morphism $\partial_{\mathfrak{A}, F^c} : K_0(\mathfrak{A}, F^c) \to \mathrm{AC}(\mathfrak{A})$ defined in §4.3, we obtain a map

$$\hat{\psi}_t : H_t^1(\mathrm{Spec}(F), G_F) \to \mathrm{AC}(\mathfrak{A}).$$

The results of [24] may also be used to give a description of $\mathrm{Im}(\hat{\psi}_t)$. It follows from Proposition 4.11 that such a description may be viewed as a realisability result for certain metrised $\mathfrak{A}$-modules (cf. Remark 6.9). □

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