MATH 115A SOLUTION SET IV
FEBRUARY 10, 2005

(1) Suppose that f and g are multiplicative functions. Prove that the function F defined by

$$F(n) = \sum_{d|n} f(d)g(n/d)$$

is also multiplicative.

Solution:

Let m and n be relatively prime positive integers. Then

$$\begin{aligned} F(mn) &= \sum_{d|mn} f(d)g(mn/d) \\ &= \sum_{\substack{d_1|m \\ d_2|n}} f(d_1d_2)g(mn/d_1d_2) \\ &= \sum_{\substack{d_1|m \\ d_2|n}} f(d_1)f(d_2)g(m/d_1)g(n/d_2) \\ &= \left(\sum_{d_1|m} f(d_1)g(m/d_1)\right)\left(\sum_{d_2|n} f(d_2)g(n/d_2)\right) \\ &= F(m)F(n). \end{aligned}$$

(2) (i) For each positive integer n, show that

$$\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0.$$

[Hint: What can you say about the four consecutive integers n, $n+1$, $n+2$ and $n+3$ modulo 4? If you find yourself doing lots of algebraic manipulations to solve this problem, then you are almost certainly on the wrong track.]

(ii) For any integer $n \geq 3$, show that

$$\sum_{k=1}^{n} \mu(k!) = 1.$$

Solution:

(i) Given any four consecutive integers n, $n+1$, $n+2$ and $n+3$, one of them will be divisible by 4. If this integer is m, say, then $\mu(m) = 0$.

(ii) For $n \geq 3$, we have

$$\begin{aligned}\sum_{k=1}^{n} \mu(k!) &= \mu(1!) + \mu(2!) + \mu(3!) + \ldots + \mu(n!) \\ &= \mu(1) + \mu(2) + \mu(6) \\ &= 1 + (-1) + 1 \\ &= 1.\end{aligned}$$

(3) The von Mangoldt function Λ is defined by

$$\Lambda(n) = \begin{cases} \log(p), & \text{if } n = p^k, \text{ where } p \text{ is a prime, and } k \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\Lambda(n) = \sum_{d|n} \mu(n/d) \log(d) = -\sum_{d|n} \mu(d) \log(d).$$

[Hint: First show that $\sum_{d|n} \Lambda(d) = \log(n)$, and then apply the Möbius inversion formula.]

Solution:

First observe that if p is prime, then

$$\begin{aligned}\sum_{d|p^k} \Lambda(d) &= \Lambda(1) + \Lambda(p) + \Lambda(p^2) + \ldots + \Lambda(p^k) \\ &= 0 + \log(p) + \log(p) + \ldots + \log(p) \\ &= k \log(p).\end{aligned}$$

Now if the prime factorisation of a positive integer n is given by $n = p_1^{k_1} \ldots p_r^{k_r}$, then the only non-zero terms in $\sum_{d|n} \Lambda(d)$ come from divisors d of the form $p_i^{s_i}$. Hence

$$\begin{aligned}\sum_{d|n} \Lambda(d) &= \sum_{i=1}^{r} \left(\sum_{d|p_i^{k_i}} \Lambda(d) \right) \\ &= \sum_{i=1}^{r} k_i \log(p_i) \\ &= \log(n).\end{aligned}$$

Applying the Möbius inversion formula yields

$$\begin{aligned}\Lambda(n) &= \sum_{d|n} \mu(d)\log(n/d)\\ &= (\log(n))\left(\sum_{d|n}\mu(d)\right) - \sum_{d|n}\mu(d)\log(d)\\ &= (\log(n))\cdot 0 - \sum_{d|n}\mu(d)\log(d)\\ &= \sum_{d|n}\mu(d)\log(d).\end{aligned}$$

(4) Let $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$ be the prime factorisation of an integer $n > 1$. If f is a multiplicative function that is not identically zero, prove that

$$\sum_{d|n}\mu(d)f(d) = (1-f(p_1))(1-f(p_2))\cdots(1-f(p_r)).$$

[Hint: Use the fact that the function F defined by $F(n) = \sum_{d|n}\mu(d)f(d)$ is multiplicative (why is this so?), and is therefore determined by its values on powers of primes.]

Solution:

If f is a multiplicative function, then so is μf. Hence the function F defined by $F(n) = \sum_{d|n}\mu(d)f(d)$ is also multiplicative (see either Problem 1 above, or a result that we proved in class). If p is a prime, then

$$\begin{aligned}F(p^k) &= \mu(1)f(1) + \mu(p)f(p) + \mu(p^2)f(p^2) + \cdots + \mu(p^k)f(p^k)\\ &= \mu(1)f(1) + \mu(p)f(p)\\ &= 1 - f(p).\end{aligned}$$

Hence, if $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$, then

$$\begin{aligned}F(n) &= F(p_1^{k_1})F(p_2^{k_2})\cdots F(p_r^{k_r})\\ &= (1-f(p_1))(1-f(p_2))\cdots(1-f(p_r)).\end{aligned}$$

(5) Let $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$ be the prime factorisation of an integer $n > 1$. Use the result of Problem 4 above to establish the following:

(a) $\sum_{m|n}\mu(m)d(m) = (-1)^r$.
(b) $\sum_{d|n}\mu(d)\sigma(d) = (-1)^r p_1p_2\cdots p_r$.
(c) $\sum_{d|n}\mu(d)/d = (1-1/p_1)(1-1/p_2)\cdots(1-1/p_r)$.
(d) $\sum_{d|n} d\mu(d) = (1-p_1)(1-p_2)\cdots(1-p_r)$.

Solution:

Suppose that $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$. Then Problem 4 above implies that

(a)

$$\begin{aligned}\sum_{m|n}\mu(m)d(m) &= (1-d(p_1))(1-d(p_2))\cdots(1-d(p_r))\\ &= (1-2)(1-2)\cdots(1-2)\\ &= (-1)^r.\end{aligned}$$

(b)

$$\begin{aligned}\sum_{d|n}\mu(d)\sigma(d) &= (1-\sigma(p_1))(1-\sigma(p_2))\cdots(1-\sigma(p_r))\\ &= (1-(1-p_1))(1-(1-p_2))\cdots(1-(1-p_r))\\ &= (-1)^r p_1p_2\cdots p_r.\end{aligned}$$

(c) If we set $f(n) = 1/n$, then

$$\begin{aligned}\sum_{d|n}\mu(d)/d = \sum_{d|n}\mu(d)f(d) &\\ &= (1-f(p_1))(1-f(p_2))\cdots(1-f(p_r))\\ &= (1-1/p_1)(1-1/p_2)\cdots(1-1/p_r)).\end{aligned}$$

(d) If we set $f(n) = n$, then

$$\begin{aligned}\sum_{d|n}d\mu(d) = \sum_{d|n}\mu(d)d &\\ &= (1-f(p_1))(1-f(p_2))\cdots(1-f(p_r))\\ &= (1-p_1)(1-p_2)\cdots(1-p_r).\end{aligned}$$

(6) Let $S(n)$ denote the number of square-free divisors of n. Show that

$$S(n) = \sum_{d|n}|\mu(d)| = 2^{\omega(n)},$$

where $\omega(n)$ is the number of distinct prime factors of n.

Solution:

Let $S(n)$ denote the number of squarefree divisors of n. If $n > 1$, and $n = p_1^{k_1}\cdots p_r^{k_r}$, then a divisor d of n will be squarefree provided that $d = p_1^{j_1}p_2^{j_2}\cdots p_r^{j_r}$, with $0 \le j_i \le 1$. There are 2^r such divisors. Hence $S(n) = 2^r$.

Next, we claim that the function $|\mu(n)|$ is multiplicative. For suppose that m and n are positive integers with $(m, n) = 1$. Then

$$\begin{aligned}
|\mu|(mn) &= |\mu(mn)| \\
&= |\mu(m)\mu(n)| \\
&= |\mu(m)||\mu(n)| \\
&= |\mu|(m)|\mu|(n).
\end{aligned}$$

This implies that the function $F(n)$ defined by $F(n) = \sum_{d|n} |\mu(d)|$ is also multiplicative. If p is a prime, then

$$\begin{aligned}
F(p^k) &= |\mu(1)| + |\mu(p)| + |\mu(p^2)| + \cdots + |\mu(p^k)| \\
&= |\mu(1)| + |\mu(p)| \\
&= 1 + |-1| \\
&= 2.
\end{aligned}$$

Hence

$$\begin{aligned}
F(n) &= F(p_1^{k_1}) \cdots F(p_r^{k_r}) \\
&= 2 \cdots 2 \\
&= 2^r \\
&= S(n).
\end{aligned}$$