1. Let \( p \) be an odd prime, and let \( \zeta_p \) be a primitive \( p \)th root of unity. Set \( \Gamma = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \), and let \( \chi : \Gamma \to \mathbb{C}^* \) be a character of order \( n > 1 \) (i.e. \( \chi \) is a group homomorphism, and \( n \) is the least integer such that \( \chi^n \) is the trivial homomorphism). We define the Gauss sum \( \tau(\chi, \zeta_p) \) by

\[
\tau(\chi, \zeta_p) = \sum_{\gamma \in \Gamma} \chi(\gamma) \zeta_p^{\gamma}.
\]

(i) Show that, for \( \gamma \in \Gamma \), we have

\[
\tau(\chi, \zeta_p^\gamma) = \chi(\gamma^{-1}) \tau(\chi, \zeta_p).
\]

(ii) Show that

\[
\tau(\chi, \zeta_p)\overline{\tau(\chi, \zeta_p)} = p.
\]

(Here \( \overline{z} \) denotes the complex conjugate of \( z \).)

(iii) Let \( \chi \) be the unique character of \( \Gamma \) of order 2. From (i) and (ii), deduce that

\[
\tau(\chi, \zeta_p) = \pm \sqrt{\left(\frac{-1}{p}\right)} p.
\]

2. Describe the factorisation of the ideals generated by 2, 3, 5 in \( \mathbb{Q}(\sqrt[3]{6}) \).

3. Let \( \theta \) satisfy \( \theta^3 - \theta - 1 = 0 \). Describe the factorisation of the ideals generated by 2, 3, 5, 23 in \( \mathbb{Q}(\theta) \).