(1) Prove that $\sqrt{n} - 1 + \sqrt{n} + 1$ is irrational for every integer $n \geq 1$.

(2) (a) Given a real number $x$ and an integer $N > 1$, prove that there exist integers $h$ and $k$ with $0 < k \leq N$ such that $|kx - h| < 1/N$. (Hint: Consider the $N + 1$ numbers $tx - \lfloor tx \rfloor$ for $t = 0, 1, ..., N$ and show that some pair differs by at most $1/N$.)

(b) If $x$ is an irrational number, prove that there are infinitely many rational numbers $h/k$ with $k > 0$ such that $|x - h/k| < 1/k^2$. (Hint: Assume that there are only a finite number $h_1/k_1, ..., h_r/k_r$. Obtain a contradiction by applying part (a) with $N > 1/\delta$, where $\delta$ is the smallest of the numbers $|x - h_i/k_i|$.)

(3) Show that the supremum and infimum of a set are uniquely defined whenever they exist.

(4) Let $A$ and $B$ be two sets of positive numbers bounded above, and let $a = \sup A$, $b = \sup B$. Let $C$ be the set of all products of the form $xy$, where $x \in A$, and $y \in B$. Prove that $ab = \sup C$. 