(1) Suppose that \( f(n) \) and \( g(n) \) are bounded sequences in \( \mathbb{R} \). Prove that
\[
\lim f(n) + \lim g(n) \leq \lim(f(n) + g(n)) \leq \lim f(n) + \lim g(n).
\]

(2) Evaluate the following limits:
(i) \( \lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n}) \).
(ii) \( \lim_{n \to \infty} (a^n + b^n)^{1/n}, \quad (a > 0, \ b > 0) \).
(iii) \( \lim_{n \to \infty} n!/n^n \).

(3) Let \((a_n)\) be a sequence of strictly positive real numbers.
(i) Prove that if \( a_n \to a \) (finite or infinite) as \( n \to \infty \), then \((a_1...a_n)^{1/n} \to a \) as \( n \to \infty \).
(ii) Deduce from part (i) that \( \lim_{n \to \infty} n^{1/n} = 1 \).
(iii) Prove that \((n!)^{1/n} \to \infty\), but that \((n!)^{1/n^2} \to 1\) as \( n \to \infty \).
[You may not use any properties of the logarithmic function in answering this question.]

(4) Let \((u_n)\) be an infinite sequence of strictly positive real numbers.
(a) Prove that if \( u_{n+1}/u_n \to l \) as \( n \to \infty \), then \( u_n^{1/n} \to l \) as \( n \to \infty \).
(b) Prove that the sequence \((n/(n!)^{1/n})\) is convergent, and find its limit as \( n \to \infty \).
(c) Construct a sequence \((u_n)\) such that \( \lim_{n \to \infty} u_n^{1/n} \) exists, but \( \lim_{n \to \infty} u_{n+1}/u_n \) does not.
[When answering this question, you may assume that \( k^{1/n} \to 1 \) for positive \( k \) and that \( [1 + (1/n)]^n \to e \) as \( n \to \infty \).

(5) (Harder) Let \( x_1, x_2, ... \) be a sequence of positive reals, and suppose that
\[
\lim_{n \to \infty} (x_1 + ... + x_n)/n = 1.
\]
Let \( 1 \leq m(n) \leq n \) be such that \( x_{m(n)} = \max\{x_1, x_2, ..., x_n\} \).
(a) Prove that \( x_n/n \to 0 \) as \( n \to \infty \).
(b) Prove that \( x_{m(n)}/n \to 0 \) as \( n \to \infty \).
(c) By comparing $x^\alpha_r$ with $x^{-\alpha+1}\alpha_{m(n)}$, show that

$$\lim_{n \to \infty} \left( x_1^\alpha + x_2^\alpha + ... + x_n^\alpha \right) / n^\alpha = 0$$

if $\alpha > 1$, and

$$\lim_{n \to \infty} \left( x_1^\alpha + x_2^\alpha + ... + x_n^\alpha \right) / n^\alpha = \infty$$

if $0 < \alpha < 1$. 

2