(1) Let \((a_n) (n = 1, 2, \ldots)\) be a decreasing sequence of positive real numbers. By using the General Principle of Convergence, or otherwise, prove that if there exists a positive number \(k\) such that \(a_n \geq k/n\) for infinitely many \(n\), then \(\sum_{n=1}^{\infty} a_n\) diverges.

Give an example of a decreasing sequence \((a_n)\) of positive numbers such that \(n a_n \to 0\) as \(n \to \infty\), and \(\sum_{n=1}^{\infty} a_n\) diverges. Show that your example has these properties.

(2) Give either a proof or a counterexample for each of the following:

(i) If the sequence \((a_n) (n = 1, 2, \ldots)\) of positive numbers tends to 0 as \(n \to \infty\), then \(\sum_{n=0}^{\infty} (-1)^n a_n^{1/n}\) converges.

(ii) If the sequence \((a_n)\) of real numbers is such that, for every positive integer \(p\), \(|a_n + p - a_n| \to 0\) as \(n \to \infty\), then \((a_n)\) converges.

(3) Let \((a_n)\) be a decreasing sequence of positive real numbers such that \(a_n \to 0\) as \(n \to \infty\). Prove that \(\sum_{n=1}^{\infty} b_n\) is convergent, where

\[b_n = \frac{a_1 + a_2 + \ldots + a_n}{n}\]

for \(n = 1, 2, \ldots\).

(4) Let \((x_n)\) be a sequence of real numbers, with \(\lim_{n \to \infty} x_n = 0\), and let \(s_n = \sum_{k=1}^{n} x_k\).

Prove that

(a) \(n^{-1}s_n \to 0\) as \(n \to \infty\);

(b) if \((x_n)\) is monotonically decreasing, and if \((nx_n)\) does not converge to zero, then \(s_n \to \infty\) as \(n \to \infty\).

Deduce that \(\sum 1/n\) diverges.

(5) Discuss the convergence of the series

\[
\sum_{n=1}^{\infty} \frac{(1 + x)(1 + 2x)\ldots(1 + nx)}{(1 + x)(2 + x)\ldots(n + x)}
\]
for all positive values of $x$. You should state precisely any tests for convergence that you use.

(6) The sequences $(a_n), (b_n)$ of real numbers are such that $a_n \neq 0$ or 1, $\sum_{n=1}^{\infty} a_n$ is convergent, and $b_n/a_n \to 1$ as $n \to \infty$. If $a_n > 0$ for all $n$, prove that the series $\sum_{n=1}^{\infty} a_n^2$, $\sum_{n=1}^{\infty} a_n/(1 - a_n)$, and $\sum_{n=1}^{\infty} b_n$ are convergent. Which, if any, of these series necessarily converges if the restriction $a_n > 0$ is lifted? Give proofs or counterexamples as appropriate.