# Math 108A Practice Midterm 1 Solutions

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2.2 True or false:

(a) *Any set containing a zero vector is linearly dependent.*
True. Let $S = \{\mathbf{0}, v_1, \ldots, v_n\}$ be a set of vectors; then

$$1 \cdot \mathbf{0} + 0 \cdot v_1 + 0 \cdot v_2 + \cdots 0 \cdot v_n = 0$$

shows that the zero vector can be written as a nontrivial linear combination of the vectors in $S$.

(b) *A basis must contain* $\mathbf{0}$.
False. A basis must be linearly *independent*; as seen in part (a), a set containing the zero vector is not linearly independent.

(c) *Subsets of linearly dependent sets are linearly dependent.*
False. Take $\mathbb{R}^2$; then $\{(1,0),(2,0)\}$ is a linearly dependent set with the linearly independent subset $\{(1,0)\}$. Alternatively, if we let $S$ be linearly independent, $S \subset S \cup \{0\}$ gives another counterexample.

(d) *Subsets of linearly independent sets are linearly independent.*
True. Suppose that $\{v_1, \ldots, v_n\}$ is linearly independent and that $\{v_1, \ldots, v_k\}$ is a subset (so that $k < n$). Furthermore, suppose that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

for some scalars $c_1, \ldots, c_k$. Then

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + 0 v_{k+1} + 0 v_{k+2} + \cdots + 0 v_n = 0$$

expresses the zero vector as a linear combination of $\{v_1, \ldots, v_n\}$; by the linear independence of $\{v_1, \ldots, v_n\}$, all of the scalars must be zero. In particular, $c_1 = c_2 = \cdots = c_k = 0$, which shows $\{v_1, \ldots, v_k\}$ to be linearly independent.

(e) *If* $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}$ *then all scalars* $\alpha_k$ *are zero.*
False. This is true exactly if $\{v_1, \ldots, v_n\}$ is a linearly independent set; for a counterexample, see the example in part (a) above.

2.3 *Recall that a matrix is symmetric if* $A = A^t$. *Write down a basis in the space of symmetric* $2 \times 2$ *matrices. How many elements are in the basis?*

Let $S = \{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\}$. We claim that $S$ is the required basis. For any scalars $a, b, c$:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

hence any symmetric matrix is a linear combination of the elements of $S$. That is, $S$ spans the set of symmetric matrices.

Suppose that a linear combination of the elements in $S$ gives the zero matrix:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies that $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}a&b\\b&c\end{smallmatrix}\right)$, so that $a = b = c = 0$. Thus $S$ is linearly independent; this with the fact that $S$ spans our space implies that $S$ is a basis, as claimed. The size of our basis, and hence the dimension of the space, is three.

2.6 *Is it possible that the vectors* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *are linearly dependent, but the vectors* $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3, \mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$ *are linearly independent?*

No. Suppose that $\{v_1, v_2, v_3\}$ is a linearly dependent set. Then one of them is a linear combination of the others; without loss of generality, we write $v_3 = av_1 + bv_2$ for some scalars $a, b$. Let's denote $V = \text{span}\{v_1, v_2\}$. Since $V$ is spanned by a set of two vectors, $\dim V \leq 2$. Notice that

$$\begin{aligned} w_1 &= v_1 + v_2 \in V \\ w_2 &= v_2 + v_3 = av_1 + (b+1)v_2 \in V \\ w_3 &= v_3 + v_1 = (a+1)v_1 + bv_2 \in V, \end{aligned}$$

so that $S = \{w_1, w_2, w_3\}$ is a set of three vectors in a space of dimension at most 2. By one of our theorems, $S$ cannot possibly be linearly independent.

1.8 *Prove that the intersection of a collection of subspaces is a subspace.*

Let $U_1, \ldots, U_n$ be a collection of subspaces and set $V = \cap_{i=1}^n U_i$. Certainly $V$ contains $\mathbf{0}$ since $\mathbf{0} \in U_i$ for each $i$, so we need only check closure of $V$ under scalar multiplication and vector addition.

Suppose that $v \in V$ and $c \in F$. Then $v \in U_i$ for each $i$, and since each $U_i$ is a subspace, $cv \in U_i$. Since $cv$ is in each $U_i$, this by definition means $cv \in V$ and $V$ is closed under scalar multiplication.

Suppose that $v, w \in V$. Then for each $i$, we have $v, w \in U_i$ and $v + w \in U_i$ since each $U_i$ is a subspace. As before, this implies $v + w \in V$, so that $V$ is closed under vector addition. All required properties hold, so $V$ is indeed a subspace.

Technical note: The wording of this problem *really* means that $\cap_{i \in A} U_i$ is a subspace for any (potentially super-uncountable) collection of subspaces $\{U_i\}_{i \in A}$. But the argument is completely identical, and most people don't think to address this point in linear algebra.