Math 108A - Home Work # 4 Solutions

LADR Problems, p. 59-60:

2. Let $f(x,y) = x^3/(x^2+y^2)$ for all $(x,y) \neq (0,0)$ and define $f(0,0) = 0$. Then $f(ax,ay) = a^3x^3/(a^2x^2+a^2y^2) = af(x,y)$ for any $a \neq 0$. If $a = 0$, $f(ax,ay) = f(0,0) = 0 = 0 \cdot f(x,y)$. To show that $f : \mathbb{R}^2 \to \mathbb{R}$ is not linear, consider $f(1,0) = 1$ and $f(0,1) = 0$, however $f((1,0)+(0,1)) = f(1,1) = 1/2 \neq 1+0$.

4. We must show that $V = null(T) + Fu$ and also that $null(T) \cap Fu = \{0\}$. First suppose, $v \in null(T) \cap Fu$. This means that $v = au$ for some $a \in F$ and $Tv = 0$. Thus $T(au) = T(v) = 0$, which implies that $aT(u) = 0$. Since $T(u) \neq 0$ by assumption, we must have $a = 0$. Thus $v = 0u = 0$, and we conclude that $null(T) \cap Fu = \{0\}$.

Now let $v \in V$. To produce a vector in $null(T)$, consider $T(v)$ and $T(u)$, which are two vectors in the one-dimensional vector space F. Hence $T(v)$ and $T(u)$ must be linearly dependent, which means that $T(v) = aT(u) = T(au)$ for some $a \in F$. Hence $T(v - au) = 0$, so $v - au \in null(T)$. We now have $v = (v - au) + au \in null(T) + Fu$. This shows that $V = null(T) + Fu$.

5. Assume that $\{v_1, \ldots, v_n\}$ is a linearly independent set of vectors in V and $T : V \to W$ is an injective linear map. If $c_1T(v_1) + \cdots c_nT(v_n) = 0$ for some $c_i \in F$, then by linearity of T, we have $T(c_1v_1 + \cdots + c_nv_n) = 0$. Since we assumed that T is injective, $c_1v_1 + \cdots + c_nv_n \in null(T) = \{0\}$, which means that $c_1v_1 + \cdots + c_nv_n = 0$. By linear independence of $\{v_1, \ldots, v_n\}$ we conclude that $c_i = 0$ for all i. This shows that $\{Tv_1, \ldots, Tv_n\}$ is linearly independent.

7. Assume that $span(v_1, \ldots, v_n) = V$ and $T : V \to W$ is a surjective linear map. If $w \in W$, there exists a $v \in V$ such that $Tv = w$. We can write $v = a_1v_1 + \cdots + a_nv_n$ since the vectors v_i span V. Now, by linearity of T, we have $w = T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n)$, which shows that w is in the span of $\{T(v_1), \ldots, T(v_n)\}$. Since w was arbitrary, we see that $span(Tv_1, \ldots, Tv_n) = W$.

9. Since $T : F^4 \to F^2$ is a linear map, by the Rank-Nullity Theorem, we know that

$$4 = \dim F^4 = \dim\ null(T) + \dim\ range(T).$$

We claim that $null(T)$ is 2-dimensional. One easily checks that it is spanned by $(5,1,0,0)$ and $(0,0,7,1)$, and these two vectors are clearly linearly independent since neither is a scalar multiple of the other. Thus $\{(5,1,0,0),(0,0,7,1)\}$ is a basis for $null(T)$ and $\dim\ null(T) = 2$. (In fact, it is only necessary to check that these 2 basis vectors span $null(T)$, so that we know $\dim\ null(T) \leq 2$.) The above equality now implies that $\dim\ range(T) = 2$, and since

$range(T)$ is a subspace of F^2, which also has dimension 2, we know that $range(T) = F^2$. Thus T is surjective.

12. First assume that there exists a surjective linear map $T : V \to W$. By the Rank-Nullity Theorem, we have

$$\dim W = \dim \ range(T) = \dim V - \dim \ null(T) \leq \dim V.$$

Conversely, assume that $\dim W \leq \dim V$. Let $\{v_1, \ldots, v_n\}$ be a basis for V and $\{w_1, \ldots, w_m\}$ a basis for W, where $m \leq n$. Now define $T : V \to W$ by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m v_m, \quad \forall \ a_1, \ldots, a_n \in F.$$

One easily checks that T is linear, and it is clear that T is surjective since its range contains all linear combinations of $w_1, \ldots, w_m$, and these vectors span W.