Math 108B - Take-Home Final

Due: 12 pm on June 11, 2008

Instructions and Rules:

- You may use your notes and the texts on this exam. In addition, you may cite any result from lecture, homework problems, or the sections of the text we have covered without proof. Just be sure to make clear which result you are using and how you are using it.
- You may not work together or talk to other people about these problems. Rena and I will not answer any questions directly related to the exam, other than for clarification. We will answer questions about material from lecture, past homework problems, etc.
- You must fully justify your answers in order to receive full credit. Partial credit will be given for work that is relevant and correct.
- Your proofs may be graded for clarity and organization, in addition to correctness.

1. (15 points) True or False? Justify your answers. In parts (a) and (b), $T \in \mathcal{L}(V)$ for a finite-dimensional $\mathbb{C}$-vector space V.
   (a) If T is diagonalizable, then so is T^2.
   (b) If T^2 is diagonalizable, then so is T.
   (c) There is an inner-product defined on $\mathcal{P}_2(\mathbb{R})$ by
   $$\langle p(x), q(x)\rangle = p(0)q(0) + p(1)q(1).$$

2. (13 points) Let A be an $n \times n$ matrix over $\mathbb{C}$. The adjoint matrix A^* of A is the conjugate transpose of A.
   (a) Show that $(A + A^*)/2$ is self-adjoint and $(A - A^*)/2$ is normal.
   (b) Explain what is wrong with the following argument: Notice
   $$A = (A + A^*)/2 + (A - A^*)/2$$
   and $(A + A^*)/2$ and $(A - A^*)/2$ are both diagonalizable by the complex spectral theorem (using (a)). Thus, in some basis, A will be a sum of two diagonal matrices, and hence diagonal. Therefore, A is diagonalizable.

3. (10 points) Let V be a finite-dimensional $\mathbb{C}$-vector space and suppose that $T \in \mathcal{L}(V)$. Show that T is nilpotent if and only if 0 is the only eigenvalue of T. (Write down a proof, even if you think this was proved in lecture.)

4. (12 points) The following matrices represent linear transformations on $\mathbb{C}^n$ with respect to the standard basis. For each matrix, find its Jordan normal form and the corresponding Jordan basis of $\mathbb{C}^n$. (Actually, everything should work over $\mathbb{R}$).

$$(a)\ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}; \qquad (b)\ \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}; \qquad (c)\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \qquad (d)\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. (10 points) Let V be a finite-dimensional $\mathbb{C}$-vector space and suppose that $T \in \mathcal{L}(V)$ satisfies $T^3 = I_V$. Show that T is diagonalizable.