Math 108B - Home Work # 1 Solutions

1. For T to have the matrix
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$
with respect to a basis $\{u_1, u_2\}$ of $\mathbb{R}^2$ and a basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$, means simply that $Tu_1 = v_1$ and $Tu_2 = v_2$. Hence $\{u_1, u_2\}$ can remain the standard basis, and then $v_1 = (1, 2, 0)$ and $v_2 = (-1, 2, 3)$ will be the columns of the given matrix for T. Since v_1 and v_2 are linearly independent, we can complete them to a basis. To do this we just need to find a third vector of $\mathbb{R}^3$ that is not a linear combination of v_1 and v_2. For instance, $v_3 = e_3 = (0, 0, 1)$ works.

2. We must multiply the given matrix on the right by the change of basis matrix C whose columns are the coordinates of the new basis w_1, w_2 in the old basis $\{v_1, v_2\}$, and we must multiply it on the left by the change of basis matrix C^{-1} whose columns are the coordinates of v_1, v_2 in the new basis $\{w_1, w_2\}$. To find C, note that $w_1 = (1, 2) = \frac{3}{2}(1, 1) + \frac{1}{2}(-1, 1) = \frac{3}{2}v_1 + \frac{1}{2}v_2$ and $w_2 = (0, 1) = ((1, 1) + (-1, 1))/2 = \frac{1}{2}v_1 + \frac{1}{2}v_2$. Hence
$$C = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$
To get C^{-1}, note $v_1 = (1, 1) = (1, 2) - (0, 1) = w_1 - w_2$ and $v_2 = (-1, 1) = -(1, 2) + 3(0, 1) = -w_1 + 3w_2$. Hence
$$C^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix},$$
and the matrix for T in the new basis is
$$\begin{aligned} C^{-1}AC &= \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 13/2 & 3/2 \\ -11/2 & 3/2 \end{pmatrix} \end{aligned}$$

3. Let $T : V \to W$ be a linear transformation, and let $\{v_1, \ldots, v_n\}$ be a basis for V. Show that T is invertible if and only if $\{Tv_1, \ldots, Tv_n\}$ is a basis for W.

   Solution. $\Leftarrow$: Suppose $\{Tv_1, \ldots, Tv_n\}$ is a basis for W, and write $w_i = Tv_i$ for each i. Then we can define a linear transformation $L : W \to V$ by
$$L(c_1 w_1 + \cdots + c_n w_n) = c_1 v_1 + \cdots c_n v_n, \quad \forall\ c_1, \ldots, c_n \in F.$$

L is well-defined since the vectors $w_1, \ldots, w_n$ are linearly independent, and since these vectors span W, L is defined for all vectors in W. Clearly $LTv_i = Lw_i = v_i$ and $TLw_i = Tv_i = w_i$ for every i. Using linearity of T and L it follows that $LTv = v$ and $TLw = w$ for all $v \in V$ and all $w \in W$. Thus L is the inverse of T and T is invertible.

$\Rightarrow$: Let $L = T^{-1}$. We first show that $Tv_1, \ldots, Tv_n$ span W. Let $w \in W$ and write $Lw = c_1v_1 + \cdots + c_nv_n$ in V. Applying T, we get

$$w = TLw = c_1Tv_1 + \cdots + c_nTv_n.$$

To show that $Tv_1, \ldots, Tv_n$ are linearly independent, suppose that $c_1Tv_1 + \cdots + c_nTv_n = 0$ for scalars c_i. Applying L, we get

$$0 = L(0) = c_1LTv_1 + \cdots + c_nLTv_n = c_1v_1 + \cdots c_nv_n.$$

Since $v_1, \ldots, v_n$ are linearly independent, we must have $c_i = 0$ for all i.

4. The **trace** of an $n \times n$ matrix A is defined as the sum of all the entries on the main diagonal of A. That is,

$$tr(A) = \sum_{i=1}^{n} A_{ii},$$

where A_{ij} denotes the entry of A in the i^{th} row and j^{th} column.

(a) Show that for any two $n \times n$ matrices A and B, $tr(AB) = tr(BA)$.

(b) Use (a) to show that if X and Y are similar matrices then $tr(X) = tr(Y)$.

Solution. (a)

$$tr(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} B_{ji}A_{ij} = \sum_{j=1}^{n}(BA)_{jj} = tr(BA).$$

(b) If X and Y are similar matrices, then $X = C^{-1}YC$ for some invertible matrix C. Thus

$$tr(X) = tr(C^{-1}(YC)) = tr((YC)C^{-1}) = tr(Y).$$

5. Let V be an inner-product space, and let W be a subspace of V. Define the **orthogonal complement** of W by

$$W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W\}.$$

Show that $W^{\perp}$ is a subspace of V.

Solution. Clearly, $0 \in W^\perp$ since $\langle 0, w\rangle = 0$ for any $w \in W$. If $v \in W^\perp$ and $a \in F$, then $av \in W^\perp$ since $\langle av, w\rangle = a\langle v, w\rangle = 0$ for any $w \in W$. Finally, if $u, v \in W^\perp$, then $u + v \in W^\perp$ since $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle = 0 + 0 = 0$ for any $w \in W$. Thus $W^\perp$ is a subspace of V.