Math 108B - Home Work # 2 Solutions

1. Let $b_1, \dots, b_n$ be positive real numbers. Check that the form

$$\langle z, w\rangle = b_1 z_1 \bar{w}_1 + \cdots b_n z_n \bar{w}_n$$

defines an inner product on F^n, where $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$. (In particular, the dot product on $\mathbb{C}^n$ is an inner product.)

Solution. We must check that $\langle z, w\rangle$ is (1) linear in z; (2) positive definite; and (3) conjugate symmetric.

(1) Let $a, c \in F$ and $z, z', w \in F^n$. Then

$$\begin{aligned} \langle az + cz', w\rangle &= \sum_{i=1}^{n} b_i (a z_i + c z_i')\overline{w}_i \\ &= \sum_{i=1}^{n} a b_i z_i \overline{w}_i + \sum_{i=1}^{n} c b_i z_i' \overline{w}_i \\ &= a\langle z, w\rangle + c\langle z', w\rangle. \end{aligned}$$

(2) Let $z \in F^n$. Then

$$\langle z, z\rangle = \sum_{i=1}^{n} b_i z_i \overline{z}_i = \sum_{i=1}^{n} b_i |z_i|^2 \geq 0,$$

since all $b_i > 0$. Furthermore, equality holds if and only if $|z_i| = 0$ for all i. That is, if and only if, $z = 0$.

(3) Let $z, w \in F^n$. Then

$$\overline{\langle w, z\rangle} = \overline{\sum_{i=1}^{n} b_i w_i \overline{z}_i} = \sum_{i=1}^{n} b_i \overline{w}_i z_i = \langle z, w\rangle.$$

2. Let V be an F-vector space with basis $\{v_1, \dots, v_n\}$, and let $B = (b_{ij})$ be the $n \times n$ matrix with entries $b_{ij} = \langle v_i, v_j\rangle \in F$. Show that

   (a) $b_{ii} > 0$ for $1 \leq i \leq n$; and

   (b) $B = \bar{B}^t$, i.e., $b_{ij} = \bar{b}_{ji}$ for all $1 \leq i, j \leq n$.

Solution. (a) By definition, $b_{ii} = \langle v_i, v_i\rangle > 0$ since the inner product is positive definite.

(b) By definition, $b_{ij} = \langle v_i, v_j\rangle = \overline{\langle v_j, v_i\rangle} = \bar{b}_{ji}$ by conjugate symmetry of the inner product.

3. Give an example of a 2×2 matrix B satisfying (a) and (b) above that does not define an inner product on F^2 with $\langle e_i, e_j \rangle = b_{ij}$ for $1 \leq i, j \leq 2$. ($\{e_1, e_2\}$ is the standard basis for F^2.) Hint: Construct the matrix B so that there is a vector v whose norm would be negative with respect to the corresponding inner product.

Solution. Suppose $B = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ for $a, d > 0$ defines an inner product $\langle -, - \rangle$. If $(x, y) \in \mathbb{R}^2$, we would then have

$$\langle (x, y), (x, y) \rangle = x^2 \langle e_1, e_1 \rangle + 2xy \langle e_1, e_2 \rangle + y^2 \langle e_2, e_2 \rangle = ax^2 + 2bxy + dy^2.$$

We can obtain a contradiction by exhibiting some $a, b, d, x, y \in \mathbb{R}$ such that the above expression is negative or zero, since that will imply that this inner product is not actually positive definite. To get an example, let $y = 1$, and solve $ax^2 + 2bx + d = 0$ for x using the quadratic formula. We get $x = (-2b + \sqrt{4b^2 - 4ad})/2a$, and this is a real number as long as $4b^2 - 4ad \geq 0$. So for instance, we may take $a = d = 1,\ b = 2$ and then $(x, y) = (-1, 1)$ would have a negative inner-product with itself.

4. 4. If $||u|| = 3, ||u + v|| = 4$, and $||u - v|| = 6$, we can solve for $||v||$ using the parallelogram identity.

$$||v||^2 = (||u + v||^2 + ||u - v||^2 - 2||u||^2)/2 = 17.$$

Thus $||v|| = \sqrt{17}$.

5. The norm $||(x, y)|| = |x| + |y|$ does not come from an inner product on $\mathbb{R}^2$, since it does not satisfy the parallelogram identity. For example, let $u = (1, 0)$ and $v = (0, 1)$ then

$$||u + v||^2 + ||u - v||^2 = 2^2 + 2^2 = 8,$$

but

$$2(||u||^2 + ||v||^2) = 2(1^2 + 1^2) = 4.$$

6. Let $u, v \in V$, where V is an inner product space over $\mathbb{R}$. We have

$$||u + v||^2 = \langle u + v, u + v \rangle = ||u||^2 + 2\langle u, v \rangle + ||v||^2,$$

and

$$||u - v||^2 = \langle u - v, u - v \rangle = ||u||^2 - 2\langle u, v \rangle + ||v||^2.$$

Thus, we can solve for $\langle u, v \rangle$ by subtracting the second equation from the first to get

$$\langle u, v \rangle = \frac{||u + v||^2 - ||u - v||^2}{4}.$$

5. (Bonus) Let $x_1, \ldots, x_n$ be positive real numbers. Prove that

$$(x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) \geq n^2.$$

(Hint: Use the Cauchy-Schwarz inequality.)

Solution. Since $x_i > 0$, we may write $x_i = a_i^2$ for real numbers $a_i > 0$. Then

$$\begin{aligned}
(x_1 + \cdots + x_n)(\frac{1}{x_1} + \cdots + \frac{1}{x_n}) &= (a_1^2 + \cdots + a_n^2)(\frac{1}{a_1^2} + \cdots + \frac{1}{a_n^2}) \\
&= ||(a_1, \ldots, a_n)||^2 \cdot ||(a_1^{-1}, \ldots, a_n^{-1})||^2 \\
&\geq |(a_1, \ldots, a_n) \cdot (a_1^{-1}, \ldots, a_n^{-1})|^2 \\
&= n^2,
\end{aligned}$$

where the inequality is by (the square of) the Cauchy-Schwarz inequality for the standard dot product in $\mathbb{R}^n$.