Math 108B - Home Work # 4 Solutions

LADR Problems p. 125

24. Notice that $\varphi(p) = p(1/2)$ is a linear functional on $\mathcal{P}_2(\mathbb{R})$. Thus we follow the idea of the proof of 6.45 to find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that $\varphi(p) = \langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$ for all p. We need an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$, which we have from homework 3: $e_1 = 1, e_2 = 2\sqrt{3}x - \sqrt{3}, e_3 = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$. Now, as in the proof of 6.45, we see that

$$\begin{aligned} q &= \varphi(e_1)e_1 + \varphi(e_2)e_2 + \varphi(e_3)e_3 \\ &= 1e_1 + 0e_2 + (-\sqrt{5}/2)e_3 \\ &= 1 - (15x^2 - 15x + 5/2) \\ &= -15x^2 + 15x - 3/2. \end{aligned}$$

27. In this product, we regard F^n as an inner product space via the dot product. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$. We have $Tz \cdot w = \sum_{i=1}^{n-1} z_i\overline{w_{i+1}} = z \cdot (w_2, \ldots, w_n, 0)$. Thus $T^*(w_1, \ldots, w_n) = (w_2, \ldots, w_n, 0)$.

29. First assume that U is invariant under T. This means that $Tu \in U$ for all $u \in U$. Let $v \in U^\perp$. If $u \in U$, $\langle u, T^*v\rangle = \langle Tu, v\rangle = 0$ since $Tu \in U$. Thus $T^*v \in U^\perp$. The same argument proves the converse, since if we replace U with $U^\perp$ we know that $(U^\perp)^\perp = U$, and if we replace T with T^*, we know that $(T^*)^* = T$.

p. 158

1. (a) To show that T is not self-adjoint it suffices to find polynomials $p(x), q(x) \in \mathcal{P}_2(\mathbb{R})$ such that $\langle Tp, q\rangle \neq \langle p, Tq\rangle$. We can choose $p(x) = 1$ and $q(x) = x$, so $Tp = 0$ and $Tq = q$. Thus $\langle Tp, q\rangle = 0$, while $\langle p, Tq\rangle = \int_0^1 x\,dx = 1/2$.

(b) This is not a contradiction because the basis $\{1, x, x^2\}$ is not an orthonormal basis. In general, an operator T is self-adjoint if and only if its matrix in any *orthonormal* basis is conjugate-symmetric.

4. First assume that P is an orthogonal projection onto a subspace U of V. Let $v, w \in V$. Then $\langle Pv, w\rangle = \langle Pv, (w - Pw) + Pw\rangle = \langle Pv, Pw\rangle = \langle Pv + (v - Pv), Pw\rangle = \langle v, Pw\rangle$, where we have used the fact that $Pv, Pw \in U$ and $v - Pv, w - Pw \in U^\perp$. This shows that $P = P^*$.

Conversely, suppose $P = P^2$ and P is self-adjoint. By the spectral theorem, we can find an orthonormal basis $\{e_1, \ldots, e_n\}$ of eigenvectors for P, and the matrix of P with respect to this basis will be diagonal. If the diagonal entries of this matrix are $d_1, \ldots, d_n \in F$, then the diagonal entries of the matrix for P^2 will be $d_1^2, \ldots, d_n^2$. Since $P = P^2$, we know that $d_i = d_i^2$ for all i. Thus each d_i is either 0 or 1. Now let U be the span of those eigenvectors

e_i for which $d_i = 1$, ie. U is the eigenspace corresponding to the eigenvalue 1 of P. Thus if $u \in U$, we have $Pu = u$. Furthermore, since $U^\perp$ is the eigenspace of 0, $v \in U^\perp$ implies $Pv = 0$. Thus $P = P_U$.

8. If T is self-adjoint, that means that $Tu \cdot v = u \cdot Tv$ for all $u, v \in \mathbb{R}^3$. For $u = (1, 2, 3)$ and $v = (2, 5, 7)$, we get $Tu \cdot v = (0, 0, 0) \cdot (2, 5, 7) = 0$, while $u \cdot Tv = (1, 2, 3) \cdot (2, 5, 7) = 33$. Thus T is not self-adjoint.

9. Suppose T is a normal operator on the complex inner product space V, and that all the eigenvalues of T are real. By the spectral theorem, there exists an orthonormal basis of V consisting of eigenvectors for T. If we write T with respect to this basis, its matrix will be diagonal with real entries (its eigenvalues) along the diagonal. Since such a matrix clearly equals its conjugate transpose, and it represents T in an orthonormal basis, T must be self-adjoint.

Conversely, suppose T is a self-adjoint operator on the complex inner product space V. By Prop. 7.1 every eigenvalue of T is real.