Math 108B - Home Work # 6 Solutions

1. If $A$ is an $n \times n$ upper-triangular matrix (i.e., $A_{ij} = 0$ for all $i > j$), show that $\det A = \prod_{i=1}^{n} A_{ii}$.

**Solution.** As done in class, we can compute the determinant of $A$ by simplifying the wedge product of the columns of $A$:

$$A_{11}e_1 \wedge (A_{12}e_1 + A_{22}e_2) \wedge \cdots \wedge (A_{1n}e_1 + \cdots + A_{nn}e_n).$$

We first distribute the wedge-products across all the sums, and then use the fact that any wedge product containing some $e_i$ twice is 0. This means that the only (possibly) nonzero term we get when we distribute is from taking $e_1$ from the first factor, and then $e_2$ from the second factor (since we can’t take $e_1$ a second time), and then $e_3$ from the third factor and so forth. Thus the above wedge product simplifies to $A_{11} \cdots A_{nn}e_1 \wedge \cdots \wedge e_n$, and by definition $\det A$ is the scalar $A_{11} \cdots A_{nn}$.

2. Let $A$ be a nilpotent $n \times n$ matrix. Show that $A$ is diagonalizable if and only if $A = 0$.

**Solution.** Clearly, if $A = 0$, then $A$ is diagonal and hence diagonalizable. Conversely, assume that $A$ is diagonalizable. This means that $A = C^{-1}DC$ for a diagonal matrix $D$ and an invertible matrix $C$. Thus $D^m = (CAC^{-1})^m = CAC^{-1}CAC^{-1} \cdots CAC^{-1} = CA^mC^{-1} = 0$. However, if the diagonal entries of $D$ are $d_1, \ldots, d_n$, then the diagonal entries of $D^m$ are just $d_1^m, \ldots, d_n^m$. Since $D^m = 0$, $d_i^m = 0$ for all $i$, and hence $d_i = 0$ for all $i$. This shows that $D = 0$ and it follows that $A = C^{-1}DC = 0$.

3. This question asks you to find some $3 \times 3$ matrices. Your answers will be non-diagonalizable, since they will each have only 2 linearly independent eigenvectors.

a) Give an example of a $3 \times 3$ matrix with only one eigenvalue (over $\mathbb{C}$), but with a 2-dimensional eigenspace. What are the generalized eigenspaces of $\mathbb{C}^3$ for your example?

b) Give an example of a $3 \times 3$ matrix with only two distinct eigenvalues (over $\mathbb{C}$), each of which has a 1-dimensional eigenspace. What are the generalized eigenspaces of $\mathbb{C}^3$ for your example?

**Solution.** Our examples will be upper-triangular matrices, since in this case we can see the eigenvalues and their multiplicities directly from the main diagonal. For (a), call the single eigenvalue $\lambda$. The matrix must then have $\lambda$ in all 3 places along the main diagonal. If we leave a 0 in the $(1,2)$-entry, we see that $e_1$ and $e_2$ are eigenvectors. We must now fill in the third column so that no additional eigenvectors involving $e_3$ arise. For instance, take the matrix

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$ 

Then the eigenspace corresponding to $\lambda$ is $null(A - \lambda I) = span(e_1, e_2)$ since $(A - \lambda I)(e_3) = e_2$. Since we only have one eigenvalue, there is only one generalized eigenspace.
Since we know that $V = \mathbb{C}^3$ is the direct sum of all the generalized eigenspaces, this one generalized eigenspace must be all of $\mathbb{C}^3$.

For (b), suppose the two eigenvalues are 1 and 2. One of these must occur with multiplicity 2, so we can suppose the entries on the main diagonal are 1, 2 and 2. By setting the (1, 2)-entry to 0, we can make $e_1$ an eigenvector with eigenvalue 1 and $e_2$ an eigenvector with eigenvalue 2. As before, if we place a 1 in the (2, 3)-entry, the eigenspace of 2 will be only 1-dimensional. Thus our matrix is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$ 

The generalized eigenspaces are $\mathbb{C}e_1$ for the eigenvalue 1 and span$(e_2, e_3)$ for the eigenvalue 2.

4. LADR Solutions (p. 188-190)

3. Suppose that $a_0v + a_1Tv + \cdots + a_{m-1}T^{m-1}v = 0$. Applying $T^{m-1}$ to this equation and noting that $T^{m-1}v \neq 0$ while $T^{m}v = 0$, we get $a_0T^{m-1}v = 0$. It follows that $a_0 = 0$. Now apply $T^{m-2}$ to the equation $a_1Tv + \cdots + a_{m-1}T^{m-1}v = 0$ to get $a_1T^{m-1}v = 0$. Hence $a_1 = 0$. Repeating in this manner we see that all the $a_i$ must be 0. Thus the vectors $v, Tv, \ldots, T^{m-1}v$ are linearly independent.

5. Suppose that $(ST)^n = 0$ for some $n \geq 0$. Then $(TS)^{n+1} = TSTS \cdots TSTS = T(ST)^nS = 0$. Hence $TS$ is also nilpotent.

10. We give a counterexample. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (y, 0)$ for all $x, y \in \mathbb{R}$. Then $null(T) = \mathbb{R}e_1 = range(T)$. Hence we cannot have $\mathbb{R}^2 = null(T) \oplus range(T)$.

11. We know that $\dim V = \dim null(T^n) + \dim range(T^n)$ so it suffices to show that $null(T^n) \cap range(T^n) = \{0\}$ (Theorem 2.18 then implies that $\dim V = \dim(null(T^n) + range(T^n))$ and hence $null(T^n) + range(T^n) = V$). Let $v \in null(T^n) \cap range(T^n)$. This means that $T^n(v) = 0$ and $v = T^n(w)$ for some $w \in V$. Then $T^{2n}(w) = 0$, which implies that $w \in null(T^{2n}) = null(T^n)$ by Proposition 8.6. Thus $v = T^n(w) = 0$. 
