Math 108B - Take-Home Midterm Solutions

1. The matrix

$$\begin{pmatrix} -2 & 11 \\ 4 & 2 \end{pmatrix}$$

represents a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the basis $\{v_1, v_2\}$ where $v_1 = (3,1)$ and $v_2 = (0,2)$. Find the matrix of T with respect to the basis $\{w_1, w_2\}$ where $w_1 = (1,1)$ and $w_2 = (-1,1)$.

Solution. We must multiply the given matrix on the right by the change of basis matrix C whose columns are the coordinates of the new basis w_1, w_2 in the old basis $\{v_1, v_2\}$, and we must multiply it on the left by the change of basis matrix C^{-1} whose columns are the coordinates of v_1, v_2 in the new basis $\{w_1, w_2\}$. To find C, note that

$$w_1 = (1,1) = \frac{1}{3}(3,1) + \frac{1}{3}(0,2) = \frac{1}{3}v_1 + \frac{1}{3}v_2$$

and

$$w_2 = (-1,1) = -\frac{1}{3}(3,1) + \frac{2}{3}(0,2) = -\frac{1}{3}v_1 + \frac{2}{3}v_2.$$

Hence

$$C = \begin{pmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{pmatrix}.$$

To get C^{-1}, note

$$v_1 = (3,1) = (2,2) - (-1,1) = 2w_1 - w_2$$

and

$$v_2 = (0,2) = (1,1) + (-1,1) = w_1 + w_2.$$

Hence

$$C^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix},$$

and the matrix for T in the new basis is

$$\begin{aligned} C^{-1}AC &= \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 11 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 16 \\ -1 & -8 \end{pmatrix} \end{aligned}$$

2. Consider the vector space $M_2(\mathbb{C})$ of all 2×2 matrices with complex entries. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$, then A^* denotes the *conjugate transpose of* A, that is the matrix

$$A^* = \bar{A}^T = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}.$$

(a) For $A, B \in M_2(\mathbb{C})$, show that $\langle A, B\rangle = tr(AB^*)$ defines an inner product.

Solution. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then

$$\langle A, B\rangle = tr(AB^*) = a\overline{a'} + b\overline{b'} + c\overline{c'} + d\overline{d'}.$$

Since this is the same formula as for the usual dot product on $\mathbb{C}^4$, we know from Lecture and Homework 2 that this is an inner product.

(b) Find an orthonormal basis for $M_2(\mathbb{C})$ with respect to this inner product.

Solution. It is clear that $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis for $M_2(\mathbb{C})$, where E_{ij} denotes the matrix with 1 in entry ij, and 0's in all other entries. If we apply the Gram-Schmidt process to this basis, nothing changes, so it must be an orthonormal basis. Of course, in performing the Gram-Schmidt process we already see that the inner product of each of these basis vectors with the others is 0 and with itself is 1. So we could also just check directly that these matrices are orthonormal. Yet another way to see this is to note that these matrices correspond to the standard basis of $\mathbb{C}^4$, which is orthonormal with respect to the usual dot product.

(c) Let $U \subseteq M_2(\mathbb{C})$ be the subspace of all matrices A with $tr(A) = 0$. Find an orthonormal basis for U and describe $U^\perp$.

Solution. A basis for U is easily seen to be $\{E_{11} - E_{22}, E_{21}, E_{12}\}$, and these are orthogonal and the last two are normal. Then we only need to rescale the first matrix to give it norm 1. Currently its norm is $\sqrt{2}$, so we get an orthonormal basis $\{E_{11}/\sqrt{2} - E_{22}/\sqrt{2}, E_{21}, E_{22}\}$ for U. We know $U^\perp$ must be one-dimensional since $\dim U^\perp = \dim V - \dim U$. Thus $U^\perp$ will consist of all scalar multiples of a single matrix in $U^\perp$. One easily sees that the identity matrix I_2 is in $U^\perp$, since if $A \in U$, $\langle A, I_2\rangle = tr(AI_2^*) = tr(A) = 0$ since $A \in U$. Hence $U^\perp$ consists of all scalar multiples of I_2, and thus of all scalar matrices in $M_2(\mathbb{C})$.

3. Let V be an inner product space. If U is a subspace of V and P_U denotes the orthogonal projection onto U, we can define the *reflection* in U to be the linear transformation $R_U : V \to V$ given by $R_U(v) = 2P_U(v) - v$ for all $v \in V$.

(a) Show that $R_U^2 = Id_V$.

(b) Show that

$$\langle R_U(v), R_U(w)\rangle = \langle v, w\rangle \text{ for all } v, w \in V.$$

Hint: Recall that $P_U(v) - v \in U^\perp$ for any $v \in V$.

Solution. (a) For $v \in V$, $R_U^2(v) = R_U(2P_U(v)-v) = 2P_U(2P_U(v)-v)-(2P_U(v)-v) = 4P_U^2(v) - 4P_U(v) + v = v$, since $P_U^2 = P_U$ for any orthogonal projection.

(b)

$$\begin{aligned}
\langle R_U(v), R_U(w)\rangle &= \langle 2P_U(v)-v, 2P_U(w)-w\rangle \\
&= \langle P_U(v)-v, 2P_U(w)-w\rangle + \langle P_U(v), 2P_U(w)-w\rangle \\
&= 2\langle P_U(v)-v, P_U(w)\rangle - \langle P_U(v)-v, w\rangle \\
&\quad + \langle P_U(v), P_U(w)-w\rangle + \langle P_U(v), P_U(w)\rangle \\
&= -\langle P_U(v)-v, w\rangle + \langle P_U(v), P_U(w)\rangle \quad \text{(by hint)} \\
&= \langle P_U(v), P_U(w)-w\rangle + \langle v, w\rangle \\
&= \langle v, w\rangle \quad \text{(again by hint)}
\end{aligned}$$

4. Let V be an inner product space, and let U and W be subspaces of U. Show that

$$(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$$

and

$$(U + W)^{\perp} = U^{\perp} \cap W^{\perp}.$$

(Hint: Use one to prove the other.)

Solution. We'll show the second identity first. We first show $(U+W)^{\perp} \subseteq U^{\perp} \cap W^{\perp}$. Let $v \in (U+W)^{\perp}$. Since $U \subseteq U+W$ and $\langle v, u\rangle = 0$ for all $u \in U+W$, $\langle v, u\rangle = 0$ for all $u \in U$. Thus $v \in U^{\perp}$. Similarly, $v \in W^{\perp}$, and hence $v \in U^{\perp} \cap W^{\perp}$. For the reverse inclusion, suppose $v \in U^{\perp} \cap W^{\perp}$, and let $u + w \in U + W$ for $u \in U$ and $w \in W$. Then $\langle v, u+w\rangle = \langle v, u\rangle + \langle v, w\rangle = 0$. Hence $v \in (U+W)^{\perp}$.

We now use the second identity to prove the first. Since the second identity is true for any pair of subspaces of V, we can replace U with $U^{\perp}$ and W with $W^{\perp}$ to get

$$(U^{\perp} + W^{\perp})^{\perp} = (U^{\perp})^{\perp} \cap (W^{\perp})^{\perp} = U \cap W$$

since $(U^{\perp})^{\perp} = U$ and similarly for W. Now take the orthogonal complement of both sides to get $U^{\perp} + W^{\perp} = (U \cap W)^{\perp}$ as desired.