Math 108B - Take-Home Midterm

Due: May 23, 2008

Instructions and Rules:

- You may use your notes and the texts on this exam. In addition, you may cite any result from lecture, homework problems, or the sections of the text we have covered without proof. Just be sure to make clear which result you are using and how you are using it.
- You may not work together or talk to other people about these problems. Rena and I will not answer any questions directly related to the exam, other than for clarification. We will answer questions about material from lecture, past homework problems, etc.
- You must fully justify your answers in order to receive full credit. Partial credit will be given for work that is relevant and correct.
- Your proofs may be graded for clarity and organization, in addition to correctness.

1. Let V be a finite-dimensional vector space. We defined the *dual space* of V as the vector space $V^* = \mathcal{L}(V, F)$ of linear functionals on V. We write V^{**} for the dual space of V^*.
   (a) For $v \in V$, define $\varphi_v : V^* \to F$ by $\varphi_v(f) = f(v)$ for all $f \in V^*$. Show that φ_v is a linear map. (Thus $\varphi_v \in V^{**}$.)
   (b) Show that the function $T : V \to V^{**}$, defined by $T(v) = \varphi_v$ for all $v \in V$, is a linear map.
   (c) Show that T, as in (b), is invertible. (Recall that, in class, we've already shown that V and V^{**} are isomorphic, i.e., they have the same dimension.)

2. We say that two inner-product spaces V and W are **isometric** if there exists an invertible isometry $T : V \to W$. Prove that two finite-dimensional inner-product spaces V and W are isometric if and only if $\dim V = \dim W$. (Hint: this is similar to Theorem 3.18 in LADR.)

3. Describe all normal $n \times n$ matrices over $\mathbb{C}$ that have only one eigenvalue.

4. Suppose that $T : V \to V$ is normal. Prove that

$$\text{null}(T^k) = \text{null}(T) \quad \text{and} \quad \text{range}(T^k) = \text{range}(T) \quad \text{for all integers } k \geq 1.$$