Math 5A - Final Exam Review Problems

Winter 2009

The exam will be cumulative, but focus more on topics covered since the second midterm: Sections 6.1 - 6.4, 6.7 and 7.1-7.2. Below is an outline of the key topics (from the new material only) and sample problems of the type you may be asked on the test. Many are similar to homework problems you have done–just remember that you will be required to show your work and/or justify your answers on the exam.

Solving 2x2 Homogeneous, Linear Systems of DEs. A homogeneous, linear 2×2 system has the form

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$

for real numbers a, b, c, d. Writing $\vec{x}$ for the vector $\begin{pmatrix} x \\ y \end{pmatrix}$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have the matrix form of the system $\vec{x}' = A\vec{x}$. To find the general solution of the system, you will need to find the eigenvalues and eigenvectors of A first. You should also be able to solve initial value problems.

- **6.2: Real Eigenvalues.** There are two different formulas you need to know: one (p. 358) for the case where A has distinct eigenvalues, and the other (p. 365) for the case where A has a single repeated eigenvalue.

  1. Find the general solution $\vec{x}(t)$ of the system $\vec{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\vec{x}$, and the unique solution that satisfies the initial condition $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
  2. Find the general solution $\vec{x}(t)$ of the system $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\vec{x}$, and the unique solution that satisfies the initial condition $\vec{x}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

- **6.3: Complex Eigenvalues.** The two linearly independent real solutions are $\vec{x}_{Re}$ and $\vec{x}_{Im}$, and you should know the formulas for both (p. 375).

  3. Find the general solution $\vec{x}(t)$ of the system $\vec{x}' = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}\vec{x}$, and the unique solution that satisfies the initial condition $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
  4. Find the general solution $\vec{x}(t)$ of the system $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\vec{x}$, and the unique solution that satisfies the initial condition $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Phase Plane Portraits for Homogeneous, Linear Systems. Stability of Equilibria. Understand the meaning of the phase plane vector field and the trajectories (2.6 and 6.1). Know the definitions of Equilibrium Points and Nullclines. Know the different Stability Classifications for Equilibria as described in Section 6.4 (p. 386, 388, 389, 392).

- **6.2, 6.4: Phase Planes for Systems with Real Eigenvalues.** Be able to sketch the Separatrix, and typical trajectories in the phase plane when the matrix has distinct real eigenvalues, including the case where one eigenvalue is 0. Identify/Draw: Attracting Nodes, Repelling Nodes, Saddle Points (p. 388, 359-60) and Star Nodes (p. 392).

5. Sketch several representative trajectories in the phase plane for the system in Problem 1 above. Draw the separatrix, and give the stability classification of the equilibrium at $(0, 0)$.
6. In Problem 2, above, is the equilibrium at $(0, 0)$ stable or unstable? Justify.

- **6.3, 6.4: Phase Planes for Systems with Complex Eigenvalues.** Know how to classify the equilibrium as an Attracting Spiral, Repelling Spiral, or Center (elliptical trajectories) (p. 389).

7. For Problems 3 and 4, above, determine the stability of the equilibrium point $(0, 0)$, and describe the behavior of the trajectories near the origin.

Solving Nonhomogeneous, Linear Systems (6.7). Know how to find particular solutions to nonhomogeneous systems $\vec{x}' = A\vec{x} + \vec{f}(t)$, using the method of undetermined coefficients and variation of parameters (6.7). You should also know that the general solution of a nonhomogeneous system has the form $\vec{x} = \vec{x}_h + \vec{x}_p$ (6.1).

8. Find a particular solution to $\vec{x}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 10 \sin t \end{pmatrix}$.
9. Find a particular solution to $\vec{x}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$, $t > 0$.

Nonlinear Systems of DE's (7.1-7.2). A nonlinear autonomous system has the form

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

(For us, $f(x, y)$ and $g(x, y)$ will always be twice-differentiable functions of x and y.) You should be able to find the Equilibrium Points of such a system: these are the points (x, y)

where $f(x, y) = g(x, y) = 0$ (7.1). You also need to know how to compute the Linearization of the system at each equilibrium point (p. 434): At the equilibrium point (x_0, y_0) this is

$$\vec{u}' = J(x_0, y_0)\vec{u}, \text{ where } J(x, y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}.$$

In general, the stability classification of the equilibrium point $(0, 0)$ of the linearized system will determine the stability classification of the equilibrium point (x_0, y_0) of the nonlinear system (See the table on p. 437).

10. Find all equilibrium points of the nonlinear system

$$\begin{array}{l} x' = -2x + 3y + xy \\ y' = -x + y - 2xy^2 \end{array},$$

and calculate the linearized system at each. Use the eigenvalues and eigenvectors of the Jacobian matrices to determine whether each equilibrium point is stable/unstable and to describe the behavior of the nearby trajectories. (For fun, you might also try to sketch the phase plane portrait.)

11. Same as 10, but for the system

$$\begin{array}{l} x' = 1 - xy \\ y' = x - y^3 \end{array},$$