Math 5B - Midterm 1 Solutions

1. (a) Find parametric equations for the line that passes through the point \((2, 0, -1)\) and is perpendicular to the plane with equation \(4x - y - 2z = 1\).

**Solution.** The direction vector for this line is \(\mathbf{v} = (4, -1, -2)\) and it must pass through the point \((2, 0, -1)\). Thus we have parametric equations \((x, y, z) = (2 + 4t, -t, -1 - 2t)\).

(b) Find the equation of the unique plane that contains the two lines, \(L_1\) and \(L_2\), whose equations are:

\[
L_1: \begin{cases} 
  x = t \\
  y = 2 - t \\
  z = 3
\end{cases}, \quad L_2: \begin{cases} 
  x = -1 + 2t \\
  y = 3 - 2t \\
  z = 3t
\end{cases}.
\]

**Solution.** Since the plane contains the two lines, their direction vectors \((1, -1, 0)\) and \((2, -2, 3)\) are parallel to the plane. Hence their cross product will be a normal vector.

\[
\mathbf{n} = (1, -1, 0) \times (2, -2, 3) = \begin{vmatrix} 
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  1 & -1 & 0 \\
  2 & -2 & 3
\end{vmatrix} = -3\mathbf{i} - 3\mathbf{j} + 0\mathbf{k} = (-3, -3, 0).
\]

To get a point in the plane, we can take any point in either line, so just set \(t = 0\) in the equations for \(L_1\) to get the point \((0, 2, 3)\). The equation for the plane is then \(-3(x - 0) - 3(y - 2) + 0(z - 3) = 0\), or more simply, \(-3x - 3y + 6 = 0\).

2. Graph at least 5 level curves of \(z = y^2/x\) (label them with the corresponding values of \(z\)), and then graph the surface \(z = y^2/x\) for \(z \geq 0\). Be sure to label your axes.

**Solution.** Setting \(z = c \neq 0\), we can solve for \(x\) to get \(x = y^2/c\). These graphs are (sideways) parabolas in the \(xy\)-plane, that get less steep as \(c\) gets large, and more steep as \(c\) approaches 0. If \(z = 0\), the level curve is \(y = 0\), or just the \(x\)-axis. It is important that these level curves all have a hole where \(x = 0\) since \(z\) is not defined there. (See the link to the level curves picture.) In graphing the whole surface, we sketch two cross sections for \(x = 1\) and \(x = 2\). These again are parabolas with equations \(z = y^2\) and \(z = y^2/2\), so they get less steep as \(x\) gets larger. Note that the \(z\)-axis is not actually part of the graph, again since \(z\) is not defined when \(x = 0\). (See the link to the surface picture.)

3. Calculate the following limits, or show that they do not exist.

(a) \(\lim_{(x,y)\to(0,0)} \frac{e^{2xy-y^2}}{x^2+y^2}\)

**Solution.** As \((x, y) \to (0, 0)\), the numerator approaches \(e^0 = 1\), since it is a continuous function of \(x\) and \(y\), while the denominator approaches 0 and is always positive. Thus the ratio approaches \(+\infty\).
(b) \[ \lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} \]

**Solution.** Convert to polar coordinates to get

\[ \lim_{r \to 0^+} 2r^3 \cos \theta \sin^2 \theta = \lim_{r \to 0^+} 2r \cos \theta \sin^2 \theta = (2 \cos \theta \sin^2 \theta) \lim_{r \to 0^+} r = 0. \]

(c) \[ \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{2x-y} \]

**Solution.** We take the limit as \((x,y)\) approaches \((0,0)\) along two different lines with equations \(y=\pm x\). First, let \((x,y)\) approach \((0,0)\) along the line \(y=0\). We get

\[ \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{2x-y} = \lim_{y \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{2}, \]

using l’Hospital’s rule for the second equality. If \((x,y)\) approaches \((0,0)\) along the line \(y=x\) instead, we get

\[ \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{2x-y} = \lim_{y \to 0} \frac{\sin(2x)}{2x} = \lim_{x \to 0} \frac{2 \cos(2x)}{1} = 2, \]

using l’Hospital’s rule for the second equality. Since we get different limits depending on the direction from which we approach \((0,0)\), the limit does not exist.

4. A function \(y = (y_1,y_2)\) is defined by \(y_1 = 3x^2 + x^2\) and \(y_2 = \frac{x_1 x_2 - 1}{x_1 + 2}\).

(a) Find the Jacobian matrix \(\left( \frac{\partial y_i}{\partial x_j} \right)\), and say where \(y\) is differentiable.

**Solution.**

\[ y_x = \left( \frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_1} \right) = \left( \frac{\partial y_1}{\partial x_1}, \frac{\partial y_1}{\partial x_2}, \frac{\partial y_2}{\partial x_1}, \frac{\partial y_2}{\partial x_2} \right) = \left( \begin{array}{cc} 6x_1 & 2x_2 \\ 2x_2 + 1/(x_1 + 2)^2 & x_1 + 2 \end{array} \right). \]

Each partial derivative appearing here is a rational function and thus continuous on its domain. The only time any of them are undefined is when \(x_1 = -2\). Thus \(y\) is differentiable on the set \(\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \neq -2\}\).

(b) Approximate \(y(-1.01,2.02)\).

**Solution.** Let \(\Delta y = y(-1.01,2.02) - y(-1,2)\). We know that \(dy\) at \((-1,2)\) approximates \(\Delta y\), so we have

\[ \Delta y \approx dy|_{(-1,2)} = y_x|_{(-1,2)} d\mathbf{x} = \left( \begin{array}{cc} -6 & 4 \\ 5 & -1 \end{array} \right) \left( \begin{array}{c} -0.01 \\ 0.02 \end{array} \right) = \left( \begin{array}{c} 0.14 \\ -0.07 \end{array} \right). \]

Here we have taken \(dx_1 = -1.01 - (-1) = -0.01\) and \(dx_2 = 2.02 - 2 = 0.02\). Thus

\(y(-1.01,2.02) = y(-1,2) + \Delta y \approx (7, -3) + (0.14, -0.07) = (7.14, -3.07).\)
5. Suppose we have functions $z(y_1, y_2, y_3) : \mathbb{R}^3 \to \mathbb{R}^2$ and $y(x_1, x_2) : \mathbb{R}^2 - \{(0, 0)\} \to \mathbb{R}^3$ given by

$$
\begin{align*}
z &= \begin{cases}
z_1 &= y_1 y_2 y_3 \\
z_2 &= y_1 y_2^2 + 2 y_2 y_3^2
\end{cases} \quad \text{and} \quad
y &= \begin{cases}
y_1 &= x_2 \cos(\pi x_1) \\
y_2 &= x_2 \sin(\pi x_1) \\
y_3 &= \ln(x_1^2 + x_2^2)
\end{cases}
\end{align*}
$$

(a) Express the Jacobian matrix $\left( \frac{\partial z_i}{\partial x_j} \right)$ of the composition $z \circ y$ as a product of two matrices (do not evaluate this product).

**Solution.** The chain rule says $z_x = z_y y_x$. So

$$
z_x = \begin{pmatrix}
y_2 y_3 & y_1 y_3 & y_1 y_2 \\
y_2^2 & 2 y_1 y_2 + 2 y_3^2 & 4 y_2 y_3
\end{pmatrix}
\begin{pmatrix}
-\pi x_2 \sin(\pi x_1) & \cos(\pi x_1) \\
\pi x_2 \cos(\pi x_1) & \sin(\pi x_1)
\end{pmatrix}
\begin{pmatrix}
\frac{2 y_1}{x_1^2 + x_2^2} \\
\frac{2 y_2}{x_1^2 + x_2^2}
\end{pmatrix}
$$

(b) Find $\left( \frac{\partial z_2}{\partial x_1} \right)_{x_2}$ at $(3, -1)$ and simplify your answer.

**Solution.** To get $\left( \frac{\partial z_2}{\partial x_1} \right)_{x_2}$, we must multiply the second row of the first matrix above by the first column of the second matrix, and then we need to evaluate at $(x_1, x_2) = (3, -1)$. We get

$$
\left( \frac{\partial z_2}{\partial x_1} \right)_{x_2} = y_2^2 (-\pi x_2 \sin(\pi x_1)) + (2 y_1 y_2 + y_3^2) (\pi x_2 \cos(\pi x_1)) + 4 y_2 y_3 (\frac{2 x_1}{x_1^2 + x_2^2}).
$$

Notice that we have $y_1(3, -1) = -\cos(3 \pi) = 1$, $y_2(3, -1) = -\sin(3 \pi) = 0$ and $y_3(3, -1) = \ln 10$. Thus plugging in $x_1 = 3$ and $x_2 = -1$ in the above, the first and last terms of the sum become 0, and we are left with $y_2^2 \pi x_2 \cos(\pi x_1) = (\ln 10)^2 \pi (-1) \cos(3 \pi) = \pi (\ln 10)^2$.

6. Suppose $f(x, y)$ and $g(x, y)$ are differentiable functions, and define $h(x, y) = f(x, y) g(x, y)$. Show that $dh = df g + f dg$.

**Solution.** By definition, $dh = h_x dx + h_y dy$. By the product rule, $h_x = \frac{\partial}{\partial x} (fg) = f_x g + f g_x$ and $h_y = \frac{\partial}{\partial y} (fg) = f_y g + f g_y$. Thus, substituting these expressions into the equation for $dh$ and rearranging the terms, we have

$$
dh = (f_x g + f g_x) dx + (f_y g + f g_y) dy = g (f_x dx + f_y dy) + f (g_x dx + g_y dy) = gdf + f dg.
$$