1. Compute \( \int_C yz \, dx + 2x \, dy - y \, dz \) where \( C \) is the straight line path from \((1, 2, 1)\) to \((-1, 3, 0)\).

**Solution.** The straight line path between the two points has parametric equations \((x, y, z) = (-2t + 1, t + 2, -t + 1)\). Thus, using the definition of line integrals, the integral becomes

\[
\int_0^1 (t + 2)(1 - t)(-2) + 2(1 - 2t) - (t + 2)(-1) \, dt = \int_0^1 2t^2 - t \, dt
\]

\[
= 2\left(\frac{t^3}{3} - \frac{t^2}{2}\right)\bigg|_0^1 = 1/6.
\]

2. Find the surface area of the surface \( S \), which is parametrized by

\[
\phi(u, v) = \begin{cases} 
  x(u, v) = u - v \\
  y(u, v) = u + v \\
  z(u, v) = uv 
\end{cases}
\]

for all \((u, v)\) with \(u^2 + v^2 \leq 1\).

**Solution.** We use the formula \( S.A. = \int\int\sqrt{EG - F^2} \, du \, dv \). Here, \( E = x_u^2 + y_u^2 + z_u^2 = 2 + v^2 \), \( F = x_u x_v + y_u y_v + z_u z_v = 1 - 1 + vu = uv \), and \( G = x_v^2 + y_v^2 + z_v^2 = 2 + u^2 \). Thus \( EG - F^2 = 4 + 2(u^2 + v^2) \), and

\[
S.A. = \int\int_{u^2+v^2 \leq 1} \sqrt{4 + 2(u^2 + v^2)} \, du \, dv
\]

\[
= \int_0^{2\pi} \int_0^1 \sqrt{4 + 2r^2} \, r \, dr \, d\theta
\]

\[
= \frac{1}{4} \int_0^{2\pi} \frac{2}{3} \left(4 + 2r^2\right)^{3/2} \bigg|_0^1 \, d\theta
\]

\[
= \frac{\pi}{3} (6^{3/2} - 4^{3/2}) = \frac{\pi}{3} (6\sqrt{6} - 8)
\]

3. Let \( S \) be the top half of the unit sphere (i.e., \( S \) is given by \( x^2 + y^2 + z^2 = 1 \) and \( z \geq 0 \)), oriented by the outer normal. Integrate

\[
\int\int_S x \, dy \, dz + y \, dz \, dx + z^2 \, dx \, dy.
\]

**Solution.** \( S \) has parametrization \( x = \cos \theta \sin \phi, \ y = \sin \theta \sin \phi \) and \( z = \cos \phi \) for \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \phi \leq \pi/2 \). Notice that this parametrization induces the inner normal on the sphere, and so we need to multiply the integral by \(-1\) when we write it in terms of the parameters \( \theta \) and \( \phi \).
We now compute the Jacobians we need (these are also the components of the normal vector).

\[
\begin{align*}
\frac{dy}{dz} &= \begin{vmatrix} \cos \theta \sin \phi & \sin \theta \cos \phi \\ 0 & -\sin \phi \end{vmatrix} = -\cos \theta \sin^2 \phi \, \frac{d\theta}{d\phi} - \sin \theta \frac{d\phi}{d\phi} \\
dz{dx} &= \begin{vmatrix} 0 & -\sin \phi \\ -\sin \theta \sin \phi & \cos \theta \cos \phi \end{vmatrix} = -\sin \theta \sin^2 \phi \, \frac{d\theta}{d\phi} - \sin \phi \frac{d\phi}{d\phi} \\
dx{dy} &= \begin{vmatrix} -\sin \theta \sin \phi & \cos \theta \cos \phi \\ \cos \theta \sin \phi & \sin \theta \cos \phi \end{vmatrix} = -\sin \phi \cos \phi \, \frac{d\theta}{d\phi} - \sin \phi \frac{d\phi}{d\phi}.
\end{align*}
\]

Thus

\[
\begin{align*}
\int \int_S x \, dy \, dz + \cdots &= -\int_0^{\pi/2} \int_0^{2\pi} -\cos^2 \theta \sin^3 \phi - \sin^2 \theta \sin^3 \phi - \sin \phi \cos^3 \phi \, d\theta \, d\phi \\
&= 2\pi \int_0^{\pi/2} (\sin^2 \phi + \cos^3 \phi) \sin \phi \, d\phi \\
&= 2\pi \int_0^{\pi/2} (1 - \cos^2 \phi + \cos^3 \phi) \sin \phi \, d\phi \\
&= -2\pi (\cos \phi - \frac{1}{3} \cos^3 \phi + \frac{1}{4} \cos^4 \phi) \bigg|_{\pi/2}^0 = \frac{11\pi}{6}.
\end{align*}
\]

4. Let \( S \) be the surface given by \( z = xy^2 - 3x^2 \) with upper normal \( \mathbf{n} \), over the square with vertices \((\pm 1, \pm 1)\) in the \( xy \)-plane. If \( \mathbf{w} = (z + 3x^2)\mathbf{i} + yz\mathbf{j} + y^2\mathbf{k} \), calculate

\[
\int \int_S \mathbf{w} \cdot \mathbf{n} \, d\sigma.
\]

Solution.

\[
\begin{align*}
\int \int_S \mathbf{w} \cdot \mathbf{n} \, d\sigma &= \int_{-1}^{1} \int_{-1}^{1} (-w_z \frac{\partial z}{\partial x} - w_y \frac{\partial z}{\partial y} + w_z) \, dx \, dy \\
&= \int_{-1}^{1} \int_{-1}^{1} -(xy^2 - 3x^2 + 3x^2)(y^2 - 6x) - y(xy^2 - 3x^2)(2xy) + y^2 \, dx \, dy \\
&= \int_{-1}^{1} \int_{-1}^{1} 6x^2y^2 - xy^4 - 2x^2y^4 + 6x^3y^2 + y^2 \, dx \, dy \\
&= \int_{-1}^{1} 6y^2 - 4y^4/3 \, dy = \frac{52}{15}.
\end{align*}
\]
5. Let $S$ be the unit sphere, $x^2 + y^2 + z^2 = 1$, oriented outward, and let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = xy^2 \mathbf{i} - xz^2 \mathbf{j} + x^2z \mathbf{k}$. Use the Divergence Theorem to compute 
\[ \int \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma. \]

Solution.
\[
\int \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int \int \int_R \text{div}(\mathbf{F}) \, dxdydz
\]
\[
= \int \int \int_R y^2 + x^2 \, dxdydz
\]
\[
= \int_0^\pi \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2 \phi) \rho^2 \sin \phi \, d\rho d\phi d\theta
\]
\[
= \int_0^\pi \int_0^{2\pi} \frac{1}{3} \sin^3 \phi \, d\phi d\theta
\]
\[
= \int_0^\pi \int_0^{2\pi} \frac{2\pi}{5} (1 - \cos^2 \phi) \sin \phi \, d\phi d\theta
\]
\[
= \int_0^\pi \int_0^{2\pi} \frac{2\pi}{5} (\phi - \frac{1}{3} \cos^3 \phi) \, d\phi d\theta
\]
\[
= \int_0^\pi \frac{2\pi}{5} (\pi + 1/3 - 0 - 1/3) = -\frac{2\pi^2}{5} - \frac{4\pi}{15}
\]

6. Let $S$ be the cone $x^2 = y^2 + z^2$, $0 \leq x \leq 2$, oriented inward (so the normal vectors point toward the $x$-axis). Use Stokes’ Theorem to calculate 
\[ \int \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma, \]
where $\mathbf{F}(x, y, z) = x^2 \mathbf{i} - z \mathbf{j} + (y^2 - z) \mathbf{k}$.

Solution. Recall $\nabla \times \mathbf{F} = \text{curl}(\mathbf{F})$, so by Stokes’ Theorem, the surface integral reduces to the line integral $\oint_C \mathbf{F}_T \, ds$ where $C$ is the boundary of $S$. Now $S$ is a cone with vertex at the origin, and so its boundary is a circle lying in the plane $x = 2$. By the right-hand rule, since the normal should be roughly in the direction of the $x$-axis, the circle $C$ should be traversed in the counterclockwise direction of the $yz$-plane. Thus $C$ has parametric equations $(x, y, z) = (2, \cos t, \sin t)$ for $0 \leq t \leq 2\pi$, and
\[
\int_C \mathbf{F}_T \, ds = \int_C x^2 \, dx - z \, dy + (y^2 - z) \, dz
\]
\[
= \int_0^{2\pi} 4(0) - \sin t(-\sin t) + (\cos^2 t - \sin t) \cos t \, dt
\]
\[
= \int_0^{2\pi} \sin^2 t + (\cos^2 t - \sin t) \cos t \, dt
\]
\[
= \int_0^{2\pi} \frac{1}{2} (1 - \cos(2t)) + (1 - \sin t - \sin^2 t) \cos t \, dt
\]
\[
= \frac{1}{2} (t - \frac{1}{2} \sin(2t)) + \sin t - \frac{1}{2} \sin^2 t - \frac{1}{3} \sin^3 t \big|_0^{2\pi} = \pi.
\]
7. Let $C$ be the curve given by $x = \sin t$, $y = \cos t$, $z = \cos(2t)$ for $0 \leq t \leq 2\pi$. Use Stokes’ Theorem to evaluate

$$\oint_C xz \, dx + y^2 \, dy + z^2 \, dz.$$

**Solution.** Since $z = \cos(2t) = \cos^2 t - \sin^2 t = y^2 - x^2$, the curve $C$ lies on the surface $z = y^2 - x^2$ (the surfaces $z = 2y^2 - 1$ or $z = 1 - 2x^2$ would also work). Since $x = \sin t$, $y = \cos t$ traces out the unit circle, the interior of $C$ lies above the unit disk in the $xy$-plane. Thus $C$ is the boundary of the surface $S$ given by the graph of $z = y^2 - x^2$ for $x^2 + y^2 \leq 1$. Furthermore, since $C$ is traversed in the clockwise direction, $S$ must be given the lower normal according to the right-hand rule (thus we multiply by $-1$). Now, using Stokes’ Theorem, we have

$$\oint_C xz \, dx + y^2 \, dy + z^2 \, dz = \int_S \text{curl}(xz \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}) \cdot \mathbf{n} \, d\sigma$$

$$= -\int_S x \, dz \, dx$$

$$= -\int_{x^2+y^2\leq1} x(2y) \, dx \, dy$$

$$= \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{1} 2xy \, dy \, dx$$

$$= \int_{-1}^{1} 2x(1 - x^2) \, dx = x^2 - x^4/2\bigg|_{-1}^{1} = 0.$$

8. Show that the integral

$$\int_{(\pi/2,0,1)}^{(\pi/2,0,1)} z^2 \cos(x + y^2) \, dx + 2yz^2 \cos(x + y^2) \, dy + 2z \sin(x + y^2) \, dz$$

is independent of path and evaluate it.

**Solution.** To show that the integral is path independent, it suffices to find a function $F(x, y, z)$ such that the integrand equals $dF$. To get $F$, integrate the $dx$ term with respect to $x$ to get $F = \int z^2 \cos(x + y^2) \, dx = z^2 \sin(x + y^2) + C(y, z)$. If we now differentiate this function with respect to $y$ and $z$ (separately), we get the other two terms of the integrand when we let $C(y, z) = 0$. Thus the integral becomes

$$\int_{(-1,1,3)}^{(\pi/2,0,1)} d(z^2 \sin(x + y^2)) = z^2 \sin(x + y^2)\bigg|_{(-1,1,3)}^{(\pi/2,0,1)}$$

$$= \sin(\pi/2) - 9 \sin 0 = 1.$$
9. Let \( u \) be the vector field

\[
\mathbf{u}(x, y, z) = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j} + z^2 \mathbf{k}
\]
on \( \mathbb{R}^3 \) minus the \( z \)-axis.

(a) Show that \( \text{curl}(u) = 0 \) on this domain.

Solution.

\[
\text{curl}(\mathbf{u}) = \left| \begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} & z^2
\end{array} \right| = 0 \mathbf{i} + 0 \mathbf{j} + \left( \frac{x^2 - y^2}{(x^2 + y^2)^2} - \frac{x^2 - y^2}{(x^2 + y^2)^2} \right) \mathbf{k} = 0.
\]

(b) Show that \( u \) is not the gradient vector field of any function \( F \) on this domain.

(Hint: Find a closed curve \( C \) with \( \oint_C \mathbf{u} \cdot d\mathbf{s} \neq 0 \).

Solution. Let \( C \) be the unit circle in the \( xy \)-plane: \( x = \cos t, \ y = \sin t, \ z = 0 \) for \( 0 \leq t \leq 2\pi \). Then

\[
\int_C \mathbf{u} \cdot d\mathbf{s} = \int_0^{2\pi} \left( \frac{\sin t}{\cos^2 t + \sin^2 t} (-\sin t) - \frac{\cos t}{\cos^2 t + \sin^2 t} (\cos t) + 0 \right) dt
\]

\[
= \int_0^{2\pi} -\left( \sin^2 t + \cos^2 t \right) dt
\]

\[
= -2\pi \neq 0.
\]

Since this integral is not zero, we know that the integral \( \int_C \mathbf{u} \cdot d\mathbf{s} \) is not path-independent in the given domain, and hence \( u \) is not a gradient vector field. (You could also prove this more directly by trying to solve for \( F(x, y, z) \) with \( \nabla F = \mathbf{u} \), and showing that no solutions exist.)