Math 8 - Midterm 1 Solutions
October 19, 2007

1. (12 pts) Consider the proposition \( R \)

“If I go surfing or take a nap, then I will not go surfing or I will not take a nap.”

(a) (2 pts) Express this proposition symbolically in terms of propositional variables \( P \) and \( Q \) and logical connectives. Be sure to say what \( P \) and \( Q \) represent.

Solution. \((P \lor Q) \Rightarrow (\sim P \lor \sim Q)\), where \( P \) is “I will go surfing.” and \( Q \) is “I will take a nap.”

(b) (6 pts) Make a truth table for your answer to (a).

Solution.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \lor Q )</th>
<th>( \sim P )</th>
<th>( \sim Q )</th>
<th>( \sim P \lor \sim Q )</th>
<th>( (P \lor Q) \Rightarrow (\sim P \lor \sim Q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

(c) (4 pts) State the converse and contrapositive of \( R \) as English sentences (without using phrases like “it is not the case that”, etc.). It may help to first write them in terms of \( P \) and \( Q \).

Solution. Converse: “If I don’t go surfing or I don’t take a nap, then I will go surfing or take a nap.” In fact, this is equivalent to the simpler statement “I will go surfing or take a nap.”

Contrapositive: First notice that the negation of the conclusion \((\sim P \lor \sim Q)\) is \( P \land Q \) by De Morgan’s law. Also by De Morgan’s law, the negation of the hypothesis \((P \lor Q)\) is \( \sim P \land \sim Q \). Thus the contrapositive says “If I go surfing and take a nap, then I will not go surfing and I will not take a nap.” Of course, it sounds absurd like this, but is equivalent to saying “I won’t go surfing and take a nap.”

2. (10 pts) Let \( P \) be the proposition

“The sum of a rational number and an irrational number is irrational.”

(a) (2 pts) Rephrase \( P \) as a conditional statement. (You may want to introduce some variables \( x, y \).)

Solution. “If \( x \) is rational and \( y \) is irrational, then \( x + y \) is irrational.”

(b) (2 pts) Express \( P \) in terms of symbols and variables only, without using words.

Solution. \([(x \in \mathbb{Q}) \land (y \notin \mathbb{Q})] \Rightarrow (x + y \notin \mathbb{Q}).\)

(c) (6 pts) Prove \( P \) is true.

Solution. We will prove \( P \) by contradiction. Assume that \( P \) is not true. This means that we have a rational number \( x \) and an irrational number \( y \) such that
$x + y$ is rational. By definition of rational, we can write $x = p/q$ and $x + y = r/s$ for integers $p, q, r, s$ with $q, s \neq 0$. Thus

$$y = (x + y) - x = \frac{r}{s} - \frac{p}{q} = \frac{rq - ps}{sq},$$

which is a rational number since $rq - ps$ and $sq$ are integers with $sq \neq 0$. However, this contradicts the assumption that $y$ is irrational.

3. (a) (4 pts) List (or otherwise describe) the elements of the set $S = \{x \in \mathbb{R} \mid 3x \in \mathbb{N}\}$.

Solution. $S$ is the set of all real numbers $x$ such that $3x$ is a natural number. Thus

$$S = \left\{ \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \ldots \right\}$$

is the set of all positive rational numbers that can be written with a 3 in the denominator.

(b) (4 pts) List (or otherwise describe) the elements of the set $T = \{1/x \mid x \in S\}$.

Solution. $T$ is the set of all reciprocals of elements of $S$. Thus

$$T = \left\{ \frac{3}{1}, \frac{3}{2}, \frac{3}{3}, \frac{3}{4}, \frac{3}{5}, \ldots \right\}$$

is the set of all positive rational numbers that can be written with a 3 in the numerator.

4. (10 pts) Prove that an integer $n$ is the product of two even integers if and only if it is a multiple of 4.

Solution. ($\Rightarrow$) Assume that $n$ is the product of two even integers $x$ and $y$. Thus $x = 2k$ and $y = 2l$ for integers $k$ and $l$. Hence $n = xy = (2k)(2l) = 4kl$ is a multiple of 4.

($\Leftarrow$) Assume that $n$ is a multiple of 4. Thus we can write $n = 4k$ for some integer $k$, and then we have $n = 2(2k)$, which is the product of two even integers, 2 and $2k$. 

2