Math 8 - Homework #4 Solutions

Fall, 2007

1. Express each of the following statements using sets. Your answers should be of the form "[something] $\in$ (or $\notin$) [some set]".

   (a) x is a nonnegative integer that is smaller than 5.

   (b) Either a or b equals 1.

   (c) Neither x nor y is 0.

   Solution. (a) $x \in \{1, 2, 3, 4\}$

   (b) $1 \in \{a, b\}$

   (c) $0 \notin \{x, y\}$

2. Describe the sets from problem 2, parts (a)-(d), on page 47 of the text in the form $\{f(x) \mid x \in S\}$, where $f(x)$ is a function, and S is some set.

   Solution. (a) $A = \{2x - 2 \mid x \in \mathbb{N}\}$

   (b) $B = \{x^2 + 1 \mid x \in \mathbb{Z}\}$

   (c) $C = \{4x - 3 \mid x \in \mathbb{N}\}$

   (d) $D = \{1/x \mid x \in \mathbb{N}\}$

3. (a) Prove that $\{2k - 1 \mid k \in \mathbb{Z}\} = \{2k + 1 \mid k \in \mathbb{Z}\}$.

   Solution. For convenience, let $A = \{2k - 1 \mid k \in \mathbb{Z}\}$ and $B = \{2k + 1 \mid k \in \mathbb{Z}\}$. We must show that for any x, $x \in A \Leftrightarrow x \in B$.

   $x \in A \Rightarrow x \in B$: Assume that $x \in A$. By the definition of A, this means that $x = 2k - 1$ for some $k \in \mathbb{Z}$. Thus $x = 2k - 1 = 2k - 2 + 1 = 2(k - 1) + 1$. Since $k - 1$ is an integer, x is also an element of B.

   $x \in A \Leftarrow x \in B$: Now assume that $x \in B$. By definition, $x = 2k + 1$ for some integer k. Thus $x = 2k + 1 = 2k + 2 - 1 = 2(k + 1) - 1$. Since $k + 1$ is an integer, x is also an element of A.

   (b) Are the sets $\{2k - 1 \mid k \in \mathbb{N}\}$ and $\{2k + 1 \mid k \in \mathbb{N}\}$ also equal? Justify your answer. (Suggestion: start listing the elements in these sets by plugging in different natural numbers for k.)

   Solution. No, these sets are not equal. The first is $\{1, 3, 5, 7, \ldots\}$, but the second is $\{3, 5, 7, \ldots\}$. The first contains 1, but the second does not.

4. (optional) In class, we wrote the set of even integers as $2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\}$. In this exercise, we explore the arithmetic of sets a little more. All sets considered here will be subsets of $\mathbb{R}$, meaning that all their elements are assumed to be real numbers.

(a) If we replace $\mathbb{Z}$ with $\mathbb{R}$ in the above example, what set do we get? In other words, describe the set $2\mathbb{R}$.

Solution. $2\mathbb{R} = \mathbb{R}$ since any real number x can be written as $2(x/2)$ and $x/2$ is another real number.

(b) Let $m, n \in \mathbb{Z}$. The set of multiples of n can be written $n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$. We can also write $m\mathbb{Z} + n\mathbb{Z} = \{mx + ny \mid x, y \in \mathbb{Z}\}$ for the set of all sums of multiples of m and n. Describe the following sets: (i) $2\mathbb{Z}+3\mathbb{Z}$; (ii) $2\mathbb{Z}+4\mathbb{Z}$; (iii) $2\mathbb{N} + 3\mathbb{N}$. (Suggestion: start by listing some elements of these sets by choosing different values for x and y in the expression $mx + ny$.)

Solution. (i) $2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$. (Since $n = 3n - 2n$ for any $n \in \mathbb{Z}$.)

(ii) $2\mathbb{Z} + 4\mathbb{Z} = 2\mathbb{Z}$.

(iii) $2\mathbb{N} + 3\mathbb{N} = \{2, 3, 4, \ldots\}$.