Math 8 - Solutions to Midterm 1 Review Problems
Fall 2007

1. Consider the proposition

   "If I have either both a fish and a cat or both a cat and a dog, then I do not have a fish or a dog."

   (a) Rewrite the proposition symbolically as a statement form in terms of statement variables P, Q and R (you may use different letters if you wish). Be sure to say which propositions are represented by your variables.

   (b) Construct a truth table for your answer to (a).

   (c) By looking at the truth table, find a simpler statement form in the same variables that is logically equivalent to this proposition. (Tricky, but good practice.)

   (d) Convert your answer to (c) back into an English sentence.

   Solution. (a) Let F be "I have a fish", let C be "I have a cat", and let D be "I have a dog". The statement then becomes

   $$P = (F \wedge C) \vee (C \wedge D)] \Rightarrow \sim (F \vee D).$$

   (b)

F	C	D	$F \wedge C$	$C \wedge D$	$(F \wedge C) \vee (F \wedge D)$	$F \vee D$	$\sim (F \vee D)$	P
T	T	T	T	T	T	T	F	F
T	T	F	T	F	T	T	F	F
T	F	T	F	F	F	T	F	T
T	F	F	F	F	F	T	F	T
F	T	T	F	T	T	T	F	F
F	T	F	F	F	F	F	T	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	T	T

   (c) Notice that P is True in all cases where C is False, and also in the case when only C is True. Thus we have $P \equiv \sim C \vee \sim (F \vee D)$. (Other variations are also possible.)

   (d) This last expression can be phrased as "Either I don't have a cat or I don't have a fish or a dog." or as "If I have a cat, then I do not have a fish or a dog."

2. **True or False.** If the statement is True, give a proof. If it is False, give a reason why or a counterexample.

   (a) $a \in \mathbb{Z} \wedge b \in \mathbb{Z} \Leftrightarrow a + b \in \mathbb{Z}$

   Solution. False. The forward implication $a \in \mathbb{Z} \wedge b \in \mathbb{Z} \Rightarrow a + b \in \mathbb{Z}$ (If a and b are integers, then $a+b$ is an integer.) is true. However, the backwards implication $a + b \in \mathbb{Z} \Rightarrow a \in \mathbb{Z} \wedge b \in \mathbb{Z}$ is False when $a = b = 1/2$.

(b) $Q \Rightarrow [P \wedge (\sim Q \vee \sim P)]$ is a tautology.

Solution. False. If we start to make a truth table, we see that the proposition is False when P and Q are both True.

(c) If a perfect square is prime, then it is divisible by all primes.

Solution. True. This statement is vacuously true, because the hypothesis is always False: A perfect square n^2 can never be prime since $n^2 = n \cdot n$ is a nontrivial factorization.

3. Consider the implication "If it rains sometimes, then nobody is happy." Write English sentences for (a) the converse; (b) the contrapositive; and (c) the negation of this statement. (Your answers should be as simple and natural as possible, so you should avoid phrases like "It is not the case that..." or "It is false that...", etc.)

   Solution. (a) "If nobody is happy, then it rains sometimes."

   (b) "If somebody is happy, then it never rains."

   (c) "It rains sometimes and somebody is happy."

4. Consider the proposition: "Every nonzero rational number is equal to a product of two irrational numbers."

   (a) Write this proposition as a conditional (If..., then...) statement.

   (b) Now write it using only symbols and no words. (You may use the symbols for the sets of rational numbers, etc. from class. You do not need to worry about quantifiers here.)

   (c) Prove this proposition.

   Solution. (a) "If x is a nonzero rational number, then $x = ab$ for two irrational numbers a and b.

   (b) $[x \in \mathbb{Q} \wedge (x \neq 0)] \Rightarrow [(x = ab) \wedge (a \notin \mathbb{Q}) \wedge (b \notin \mathbb{Q})]$

   (c) Proof: Assume that x is a nonzero rational number. Then $x = \sqrt{2} \cdot (x/\sqrt{2})$. We proved in class that $\sqrt{2}$ is irrational, so we only need to show that $x/\sqrt{2}$ is also irrational. We now prove that $x/\sqrt{2}$ is irrational by contradiction. Assume that $x/\sqrt{2}$ is rational. This means that $x/\sqrt{2} = m/n$ for two *nonzero* integers m and n. If we solve for $\sqrt{2}$, we have $\sqrt{2} = nx/m$. Since $x \in \mathbb{Q}$, $x = p/q$ for two integers p and q with $q \neq 0$. Thus $\sqrt{2} = np/mq$ is a rational number (notice $mq \neq 0$). But this contradicts the fact that $\sqrt{2}$ is irrational. Hence, we can conclude that $x/\sqrt{2}$ is also irrational.

5. In this problem x and y are real numbers. Consider the proposition: "If the product xy is irrational, then either x or y is irrational."

   (a) State the contrapositive of this proposition.

   (b) Prove this proposition.

(c) Is the converse of this proposition also true (for all real numbers x, y)? Explain.

Solution. (a) The conclusion can be written $\sim (x \in \mathbb{Q}) \vee \sim (y \in \mathbb{Q})$. By De Morgan's law, this is equivalent to $\sim (x \in \mathbb{Q} \wedge y \in \mathbb{Q})$. Thus the negation of the conclusion is "x and y are rational." The negation of the hypothesis is "xy is rational." Thus the contrapositive says "If x and y are rational, then xy is rational."

(b) We prove it indirectly by proving the contrapositive stated in (a). Assume x and y are rational. Then $x = a/b$ and $y = c/d$ for integers a, b, c, d with $b, d \neq 0$. Thus $xy = ac/bd$ is rational, since ac and bd are both integers and $bd \neq 0$.

(c) The converse says that "If x or y is irrational, then xy is irrational." This is obviously false, as demonstrated by the last exercise, or the easy example $0 \cdot \sqrt{2} = 0$, which works since $x = xy = 0$ is rational, but $y = \sqrt{2}$ is not.

6. Which (if any) of the sets $A = \{1, 2, \{4, 3\}, 5\}$, $B = \{\{3, 4\}, 5, 1, 2\}$, $C = \{\{1, 2\}, 4, 3, 5\}$ are equal to each other?

   Solution. Sets A and B are equal, since they have the same elements $1, 2, \{3, 4\}$ and 5(the order they're written in does not matter, and for the same reason $\{3, 4\} = \{4, 3\}$). However, $C \neq A$ since $4 \in C$, but $4 \notin A$.

7. Express the following sets in set-builder notation, i.e., $A = \{x \in S \mid P(x)\ \}$.

   (a) $A = \{0, 1, 4, 9, 16, \ldots\}$ (ie., the set of perfect squares).
   Solution. $A = \{x \in \mathbb{Z} \mid \sqrt{x} \in \mathbb{Z}\ \}$

   (b) B is the interval $(-1, 2]$ on the real number line.
   Solution. $B = \{x \in \mathbb{R} \mid -1 < x \leq 2\}$

   (c) $C = \{9, 19, 29, \ldots\}$ is the set of all positive integers that end in 9. (This can be done without using any words.)
   Solution. $C = \{n \in \mathbb{N} \mid 10|(n+1)\ \}$ (Recall, $10|(n+1)$ means that $n+1$ is a multiple of 10.)

8. Prove that the two sets $A = \{x \in \mathbb{R} \mid x^2 \in \mathbb{Q}\}$ and

$$B = \{x/y \mid x, y \in \mathbb{R} \wedge y \neq 0 \wedge x^2, y^2 \in \mathbb{Z}\ \}$$

   are equal.

   Solution. We must prove that $z \in A \Leftrightarrow z \in B$ is true for any real number z. For the forward direction ($z \in A \Rightarrow z \in B$), assume that $z \in A$. This means that $z^2 \in \mathbb{Q}$. We also know that $z^2 \geq 0$. Hence we can write $z^2 = a/b$ for two nonnegative integers a and b with $b \neq 0$. Solving for z, we get $z = \pm\sqrt{a}/\sqrt{b}$. Since $\pm\sqrt{a}$ and $\sqrt{b}$ are real numbers whose squares are integers, $z \in B$ by definition of B.

   For the backwards direction ($z \in A \Leftarrow z \in B$), assume that $z \in B$. This means that $z = x/y$ for two real numbers x and y such that $y \neq 0$, and x^2 and y^2 are integers. Thus $z^2 = x^2/y^2$ is a ratio of two integers, and is thus rational. Hence $z \in A$ by definition of A.