Math 8, Midterm 2 Review

Fall 2007

The problems on the midterm will be similar to the ones from the homework assignments, lectures, and this review sheet. As usual, you will be required to show your work and fully justify your answers to receive full credit. The test will focus on sections 2.1, 2.3-2.8, and parts of 3.1-3.3 from the text, but will also include material from the beginning of the course. Below is a summary of the topics and types of problems you should be able to do, along with some practice problems. Solutions to these will be available later.

- **Sets: Definitions and Notation (2.1)**
  - Know the meanings and correct usage of the symbols $\in, \notin, \{\cdots \mid \cdots\}, \emptyset$: For example, you should be able to express propositions in this notation.
  - Know how to read and write both variations of Set-Builder notation that we discussed:
    1) $\{x \in S \mid P(x)\ \}$ defines the set of exactly those elements of S that make $P(x)$ true.
    2) $\{f(x) \mid x \in S\ \}$ defines the set of all elements of the form $f(x)$ where x is any element of S.
  - Set Equality. $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$. You should be able to prove that two sets are equal by using this definition.

1. Write the following sets in the form a) $\{x \in S \mid P(x)\ \}$ and the form b) $\{f(x) \mid x \in S\ \}$.
   (i) $\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots\}$
   (ii) $\{11, 21, 31, 41, \ldots\}$
2. List (or otherwise describe) the elements of the set $\{5x - 1 \mid x \in \mathbb{Z}\}$.
3. Prove that $\{5x - 1 \mid x \in \mathbb{Q}\} = \mathbb{Q}$.

- **Quantifier Notation (2.3)**
  - Know the meanings and correct usage of $\exists$ = "There exists" and $\forall$ = "for all". You should be able to write propositions using this notation.
  - De Morgan's laws for negations of quantifiers: $\sim(\exists x P(x)) \equiv \forall x \sim P(x)$ and $\sim(\forall x P(x)) \equiv \exists x \sim P(x)$.
  - You should be able to decide whether propositions involving quantifiers are true or false, and give brief justifications.

4. True or False? Give brief justifications. (The universe of discourse is $\mathbb{R}$.)
   (a) $\forall x\ \forall y\ (xy \geq 0)$
   (b) $\exists a\ \forall b\ (ba = b/a)$

5. Let $P(x, y)$ stand for the proposition "x and y are friends", and assume the universe of discourse is the set of all people. Express the following propositions symbolically:
   (a) "Every two people have a common friend."
   (b) "Nobody is friends with everyone."

- **Set Inclusion (2.4)**
  - Know the meaning and usage of the symbols $\subseteq$, $\subset$. You should be able to express propositions symbolically using this notation.
  - Know the definitions of subsets and proper subsets:
    1) A is a subset of $B \Leftrightarrow A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$.
    2) A is a proper subset of $B \Leftrightarrow A \subset B \Leftrightarrow (A \subseteq B) \wedge (A \neq B)$.
    You should be able to prove, using these definitions that one set is a subset of another.
  - You should be able to prove set equality by using the identity
    $$A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A).$$
    (You may want to try problems 3, 7 and 8 this way.)

- **Set Operations: Union, Intersection, Complement (2.5)**
  - Know the definitions of $A \cup B$, $A \cap B$, $B - A$, A'. Be able to illustrate sets obtained by combinations of these operations on sets A, B, C using a Venn diagram.
  - Theorem 2.26 (p. 65) may be helpful to know, but more important is that you should be able to prove these statements using the definitions.
  - Theorem 2.27 (p. 65) is also helpful, especially if you need to show that two sets are equal by showing that each one is a subset of the other.

6. Shade in the region corresponding to $S = A \cap (B \cup C)$ on a Venn diagram. On a separate diagram, shade in the region corresponding to $T = A \cup (B \cap C)$. If these are different, give an example of sets A, B, C and an element x that belongs to one of the sets S, T but not the other.
7. Let A and B be sets. Prove that $A \cap B = A \cup B$ if and only if $A = B$.

- **Indexed Families of Sets (2.6)**
  - Understand the definition: If I is a set, a **family of sets indexed by** I, is a collection of sets A_i, one for each element $i \in I$. This may also be written $\{A_i \mid i \in I\}$ or $\{A_i\}_{i \in I}$.
  - Know Definition 2.30 (p. 71-72) for the union $\bigcup_{i \in I} A_i$ and intersection $\bigcap_{i \in I} A_i$ of an indexed family of sets $\{A_i\}_{i \in I}$.

8. For each $i \in \mathbb{N}$, let $A_i = \{x \in \mathbb{N} \mid x \geq i\}$. Compute $\bigcup_{i\in\mathbb{N}} A_i$ and $\bigcap_{i\in\mathbb{N}} A_i$. (Give brief justifications for your answers, but not rigorous proofs.)

- **The Power Set (2.7)**
  - Know the definition of the power set of a set: For a set A, the power set of A is the set $\mathcal{P}(A) = \{X \mid X \subseteq A\}$. This is the set of all subsets of A. This can also be expressed
  $$X \subseteq A \Leftrightarrow X \in \mathcal{P}(A).$$
  You should be able to compute the power sets of some small sets.

9. What is $\mathcal{P}(\{a, \{b, c\}\})$?

10.* Is it always true that $A \cap \mathcal{P}(A) = \emptyset$? Provide a justification or a counterexample.

- **Ordered Pairs and Cartesian Products (2.8)**
  - Know the definition of the Cartesian Product of two sets A and B:
  $$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$
  - Know how to work with ordered pairs: namely, understand that
  $$(a, b) = (c, d) \Leftrightarrow (a = c \wedge b = d).$$
  Be able to use these definitions to prove that one cartesian product is a subset of (is equal to) another.

11. (a) Give an example of a subset of $\{a, b, c\} \times \{a, b, c\}$ that is not equal to $A \times B$ for any subsets A, B of $\{a, b, c\}$.
(b) Give an example of a subset of the plane $\mathbb{R}^2$ (a picture will suffice) that is not of the form $A \times B$ for subsets A, B of $\mathbb{R}$. Justify briefly.

- **Functions (3.1-3.2)**
  - Know the definition of a function $f : A \to B$, and various ways of describing a function, such as by tables or by diagrams as on p. 121.
  - Know the definition of Surjections, Injections, Bijections (and related terms). Be able to recognize when a function is surjective/injective/bijective and give examples of functions that are (or are not) surjective/injective/bijective.
  - Know the definition of Compositions of functions (3.3), and be able to compute these and give examples.
  - **Note:** The exam WILL NOT include any proofs involving functions. However, you may be asked to give a brief explanation of why a particular function is not surjective, etc.

12. Give examples of sets $A,\ B,\ C$ and functions $f : A \to B$ and $g : B \to C$ such that (Note: each letter represents a separate problem.)

    (a) f is surjective, but not bijective.

    (b) g is surjective, but $g \circ f$ is not surjective.

    (c) $g \circ f$ is bijective, but f is not bijective.

    You may give your answers in the form of tables, graphs or function diagrams, but be sure to label everything appropriately.