Name: Solution

Perm No.: ________________

Math 8 - Final Exam

June 10, 2009

Instructions:

- This exam consists of 8 problems, each worth 10 points, and one bonus problem worth up to 5 points.
- You must show all your work and fully justify your answers in order to recieve full credit. Partial credit will be given for work that is correct and relevant. Your proofs will be graded for clarity and organization, in addition to correctness.
- No books, notes or calculators are allowed.
- Write your answers on the test itself, in the space alotted. You may attach additional pages if necessary.

1	
2	
3	
4	
5	
6	
7	
8	
Total	
Bonus	

1. Write each of the following sets in set-builder notation $\{x \in U \mid P(x)\}$, where U is a set and $P(x)$ is a proposition depending on x.

(a) $A = \{1, 5, 9, 13, 17, \ldots\}$

$A = \{4k-3 \mid k \in \mathbb{N}\}$

$$\boxed{\begin{aligned} &= \{x \in \mathbb{N} \mid \exists k \in \mathbb{N}\ x = 4k-3\} \\ &= \{x \in \mathbb{N} \mid 4 \mid x+3\}. \end{aligned}}$$

(b) $B = \{0, 1/2, 2/3, 3/4, 4/5, 5/6, 6/7, \ldots\}$

$B = \{\frac{n-1}{n} \mid n \in \mathbb{N}\}$.

$$\boxed{\begin{aligned} &= \{x \in \mathbb{Q} \mid \exists n \in \mathbb{N}\ x = \tfrac{n-1}{n}\}. \\ &= \{x \in \mathbb{Q} \mid \tfrac{1}{1-x} \in \mathbb{N}\}. \end{aligned}}$$

2. Are the following propositions True or False? Give brief justifications for your answers. (The universe of discourse for all variables is $\mathbb{Z}$.)

(a) $\forall x\ \exists y\ \exists z\ (x = 2y + 4z)$.

$\boxed{\text{False.}}$ It fails for $x = 1$ (or any odd integer) since $2y+4z$ is always even & can never equal x.

(b) $\exists x\ \forall y\ (6x - y > 0)$.

$\boxed{\text{False.}}$ If an x existed such that $6x - y > 0\ \forall y$ we would have $x > y/6\ \forall y$. which is impossible.

3. Write the following statements using symbols only, and no words. (You do not have to prove them – they may not even be true.)

(a) "Every integer is equal to a sum of two perfect squares."

$\forall n \in \mathbb{Z}\ \exists a \in \mathbb{Z}\ \exists b \in \mathbb{Z}\ (n = a^2 + b^2)$

(b) "The product of any integer and any rational number is a positive real number."

$\forall n \in \mathbb{Z}\ \forall x \in \mathbb{Q} (nx > 0\ \wedge\ nx \in \mathbb{R})$

4. Determine if the following functions are one-to-one or onto, or both or neither. Justify your answers.

(a) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is defined by $f(a, b) = (b, a)$ for all $a, b \in \mathbb{N}$.

$\boxed{f \text{ is one-to-one}}$ If $f(a,b) = f(a',b')$

then $(b,a) = (b',a')$

$\Rightarrow b = b'$ & $a = a'$

$\Rightarrow (a,b) = (a',b')$.

$\boxed{f \text{ is onto}}$. Let $(a,b) \in \mathbb{N} \times \mathbb{N}$.

Then $(a,b) = f(b,a) \in \text{range}(f)$

(b) $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ is defined by $g(a, b) = a/b$. Here, $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ is the set of positive rational numbers.

$\boxed{g \text{ is not one-to-one:}}$

$g(2,1) = \frac{2}{1} = 2 = \frac{4}{2} = g(4,2)$

but $(2,1) \neq (4,2)$.

$\boxed{g \text{ is onto:}}$ If $x \in \mathbb{Q}^+$ then $x = a/b$

for $a,b \in \mathbb{N}$.

$\Rightarrow x = a/b = g(a,b) \in \text{range}(g)$.

w/ $(a,b) \in \mathbb{N} \times \mathbb{N}$.

5. Prove that for any integer $n \geq 1$

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2.$$

Induction on $n \geq 1$:

Basis Step: $n = 1$: $1 \cdot 2^1 = 2 = (1-1)2^2 + 2$ ✓

Inductive Step Assume for some $n \geq 1$

that $1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2$.

Then $\underbrace{1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n} + (n+1) \cdot 2^{n+1}$

$= (n-1) \cdot 2^{n+1} + 2 + (n+1)2^{n+1}$ by Ind. Hyp.

$= [n-1+n+1] \cdot 2^{n+1} + 2$

$= 2n \cdot 2^{n+1} + 2$

$= (n+1-1) \cdot 2^{(n+1)+1} + 2$.

as required.

Thus the formula holds $\forall\, n \geq 1$
by Induction.

6 Consider the relation $\sim$ on $\mathbb{Z}$, defined by

$$a \sim b \Leftrightarrow \exists\, n \in \mathbb{Z}\ (a = 2^n b)$$

for any $a, b \in \mathbb{Z}$. (Don't forget: $n \in \mathbb{Z}$ can be negative here.)

(a) Show that $\sim$ defines an equivalence relation on $\mathbb{Z}$.

Reflexive: $a \sim a \iff \underbrace{\exists n \in \mathbb{Z}\ (a = 2^n a)}$
True - take $n = 0$.

Symmetric $a \sim b \iff \exists n \in \mathbb{Z}\ (a = 2^n b)$
$\iff \exists n \in \mathbb{Z}\ (b = 2^{-n} a)$
$\iff \exists m \in \mathbb{Z}\ (b = 2^m a)$ $(m = -n)$
$\iff b \sim a$.

Transitive Suppose $a \sim b$ & $b \sim c$.
Then $\exists n, m \in \mathbb{Z}$ s.t. $a = 2^n b$ & $b = 2^m c$
Then $a = 2^n \cdot 2^m c = 2^{n+m} c$
& $n+m \in \mathbb{Z} \Rightarrow a \sim c$.

(b) Describe the equivalence classes of 1 and 3.

$1/\sim = \{x \in \mathbb{Z} \mid 1 \sim x\} = \{x \in \mathbb{Z} \mid \exists n \in \mathbb{Z}\ 1 = 2^n x\}$
$= \{x \in \mathbb{Z} \mid \exists m \in \mathbb{Z}\ 2^m \cdot 1 = x\}$.
$= \{2^m \mid m \in \mathbb{Z}\} \cap \mathbb{N} = \{1, 2, 4, 8, \dots\}$.

$3/\sim = \{x \in \mathbb{Z} \mid x \sim 3\} = \{x \in \mathbb{Z} \mid \exists n \in \mathbb{Z}\ x = 2^n \cdot 3\}$
$= \{2^n \cdot 3 \mid n \in \mathbb{Z}\} \cap \mathbb{N} = \{3, 6, 12, 24, 48, \dots\}$.

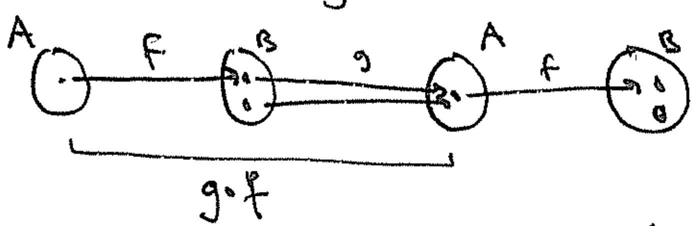
7. Let A and B be sets, and let $f : A \to B$ and $g : B \to A$ be functions such that $g \circ f$ is bijective. Is it always true that $f \circ g$ is also bijective? Give a proof or else find a counterexample.

No. If $g \circ f$ is bijective, then $f \circ g$ might not be bijective.

Counterexample: Let $A = \{a\}$ & $B = \{b, c\}$.
Define $f : A \to B$ by $f(a) = b$.
& $g : B \to A$ by $g(b) = g(c) = a$.

Then $g \circ f : A \to A$ is given by
$$(g \circ f)(a) = g(f(a)) = g(b) = a.$$

while $f \circ g : B \to B$ is given by
$$(f \circ g)(b) = f(g(b)) = f(a) = b$$
& $$(f \circ g)(c) = f(g(c)) = f(a) = b.$$
$\Rightarrow$ $f \circ g$ is not 1-to-1 & not bijective.



$f \circ g$ = not 1-to-1 or onto.

8. For two subsets A, B of a set U, define a new set

$$A \otimes B = (A \cup B) - (A \cap B).$$

Prove that $A \otimes B = \tilde{A} \otimes \tilde{B}$. (Recall $\tilde{A} = U - A$ is the complement of A.)

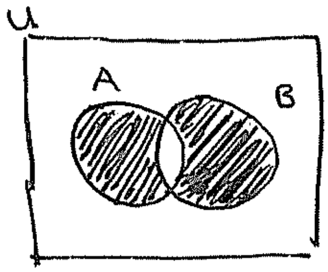
(Hint: Find a propositional form $P(x)$ such that $A \otimes B = \{x \mid P(x)\}$. Venn diagrams may also help.)

$$A \otimes B = A \cup B - (A \cap B) = \{x \in U \mid (x \in A \vee x \in B) \wedge \sim(x \in A \wedge x \in B)\}.$$

$$= \{x \in U \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}. \quad (*)$$

$$= \{x \in U \mid x \in A - B \vee x \in B - A\}$$

$$= (A - B) \cup (B - A)$$



$A \otimes B$.

So $A \otimes B$ consists of those $x \in U$ that belong to <u>exactly one</u> of A & B.

Let's use Line $(*)$ to write out $\widehat{A} \otimes \widehat{B}$

$$\widehat{A} \otimes \widehat{B} = \{x \in U \mid (x \in \widehat{A} \wedge x \notin \widehat{B}) \vee (x \in \widehat{B} \wedge x \notin \widehat{A})\}.$$

$$= \{x \in U \mid (x \notin A \wedge x \in B) \vee (x \notin B \wedge x \in A)\}.$$

$$= A \otimes B \text{ again by } (*).$$

9. **Extra Credit. (10 pts)** Is there a positive integer n such that when the last digit is moved to the beginning (eg., 3421 would become 1342) it produces $2n$? What about $3n$? $7n$? (Find an example or prove that none exists.)

See following page for answers & an algorithm.

Let n be such a number.

Let $m = 2, 3, 7$.

Let a = last digit of n

Let r = # of digits of n.

moving the last digit of n to the front yields the # $\frac{n-a}{10} + a \cdot 10^{r-1}$

So we must solve $\frac{n-a}{10} + a \cdot 10^{r-1} = m \cdot n$

$\Rightarrow a \cdot (10^r - 1) = (10m - 1)n.$

$\Rightarrow n = \underbrace{\frac{a(10^r-1)}{10m-1}}_{\text{and we need this to be an integer.}}$

ie. we need $10m-1 \mid 10^r - 1$.

we find r by looking for a power of 10 that is 1 more than a multiple of $10m-1$.

eg when $m=2$, we want $10^r \equiv 1 \pmod{19}$

note $10^2 = 100 \equiv 5 \pmod{19}$

$10^6 \equiv 5^3 \equiv 125 \equiv 11 \pmod{19}$

$10^{18} \equiv 11^3 \equiv 121 \cdot 11 \equiv 77 \equiv 1 \pmod{19}$

$\Rightarrow r = 18$ & $\boxed{n = a\left(\frac{10^{18}-1}{19}\right)}$

For n to have r digits, we need $\boxed{a \geq 2.}$

$m=3$: $\boxed{n = a\left(\frac{10^{28}-1}{29}\right)}$ $\boxed{a \geq 3}$ For $m=7$: $\boxed{n = a\left(\frac{10^{22}-1}{69}\right) \quad a \geq 7.}$

Extra Credit

Observing the pattern [illegible] see that each digit must [illegible] be [illegible] the following digit multiplied by any [illegible]. As such, we need simply start with the appropriate ending digit [illegible] given the desired integer

Consider

$$\begin{array}{r} 157894736842105263 \\ \times\quad 2 \\ \hline 315{,}769{,}473{,}684{,}210{,}526 \end{array}$$

(18 digits)

$$\begin{array}{r} 2068965517241379310344827586 \\ \times\quad 3 \\ \hline 6{,}206{,}896{,}551{,}724{,}137{,}931{,}034{,}482{,}758 \end{array}$$

(28 digits)

$$\begin{array}{r} 1014492753623188405797 \\ \times\quad 7 \\ \hline 7{,}101{,}449{,}275{,}362{,}318{,}840{,}579 \end{array}$$

(22 digits)