Math 8 - Final Exam Review Problems
Spring 2009

While the final exam will focus on topics covered since the midterm, you will still be expected to know the remaining material from chapters 1 and 2 as well. In particular, you should be able to:

- Understand and use quantifier notation and how to prove statements involving quantifiers. (Ch. 1.3, 1.6)
- Understand and Prove conditional and biconditional statements (using direct and/or indirect proofs). (Ch. 1.4 - 1.5)
- SET-BUILDER NOTATION ! Understand definitions of sets, and related notation (eg. $\in, \subseteq$, $\{x \in A \mid P(x)\}$, etc. Ch. 2.1)
- Prove that one set is a subset of another, or that two sets are equal. (Ch. 2.1-2.2)
- Know the definitions of Power Set, Union, Intersection, Set Difference, and Complements. (Ch. 2.1-2.2)
- Know the definitions of Union and Intersection of an indexed family of Sets. (Ch. 2.3).

New Topics The sections covered since the midterm are Ch. 2.4-2.5, 3.1-3.2, 3.4, 4.1-4.3, 5.1-5.3. You are not responsible for everything here, but only what is covered by the assigned readings on the webpage and by lecture.

1. **Induction (Ch. 2.4-2.5).** Focus on the basic application of the Principle of Mathematical Induction to proving $\forall\, n \in \mathbb{N}\ P(n)$, as on the top of p. 100 – This is usually enough to do most induction problems.

   The Principle of Complete (Strong) Induction (2.5) and the Well-Ordering Property (2.5) are useful generalizations.

   (a) Prove that for any integer $n \geq 1$,
   $$(x-1)(x^n + x^{n-1} + \cdots + x + 1) = x^{n+1} - 1.$$

   (b) Prove that for any integer $n \geq 1$,
   $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

2. **Relations (Ch. 3).**

   - Know the definition of Cartesian Products $A \times B$ of sets (3.1).

- Know the definitions of Reflexive, Symmetric, Antisymmetric, and Transitive (3.2); and be able to verify whether a given relation has these properties or not. You should also be able to give examples of relations that satisfy some subset of these properties. Pictures, such as directed graphs, are helpful for this purpose.
- Be able to show that a relation is an Equivalence Relation (3.2), and describe the equivalence classes.
- Be able to show that a relation is a Partial Order (beginning of 3.4).

(a) Consider the relation $\sim$ on the power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of natural numbers, defined by
$$A \sim B \Leftrightarrow A \cup B = \mathbb{N},$$
for subsets $A, B \subseteq \mathbb{N}$. Is $\sim$ an equivalence relation? Justify your answer. If so, describe the equivalence classes.

(b) Consider the relation $\approx$ on the power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of natural numbers, defined by
$$A \approx B \Leftrightarrow \min A = \min B,$$
for subsets $A, B \subseteq \mathbb{N}$. Here, $\min A$ denotes the smallest element of A, or $\min A = 0$ if $A = \emptyset$. Is $\approx$ an equivalence relation? Justify your answer. If so, describe the equivalence classes.

(c) Define a relation R on $\mathbb{N} \times \mathbb{N}$ by
$$(a, b)R(c, d) \Leftrightarrow a + b \leq c + d.$$
Is R a partial order? Which of the properties Reflexive, Symmetric, Antisymmetric, Transitive hold for R?

3. **Functions (Ch. 4).**

- You should know what a function $f : A \to B$ is: The definition in terms of relations (4.1) is not too important, but you should know that a function $f : A \to B$ is an assignment of one output in B to each input from A.
- You should be familiar with different ways of describing/defining functions, such as by directed graphs with one arrow leaving each element a of A and pointing to the element $f(a) \in B$; or by tables listing the elements of A in one column and the corresponding outputs in the other column; or by a relation/graph listing or showing all ordered pairs $(a, f(a))$; or else by a formula $f(a) = a^2 \ldots$.
- Know the definition of the inverse f^{-1} of a function f, and of the composition $g \circ f$ of two functions f and g (4.2).
- Know the definitions of one-to-one / injective, onto / surjective, and bijective (4.3). Be able to give examples of functions that have some of these properties. Be able to prove that a given function satisfies these properties or doesn't.

Give examples of the following, or explain why no example exists.

(a) A one-to-one function $f : \mathbb{N} \to \mathbb{N}$ that is not onto.

(b) An onto function $f : \mathbb{R} \to \mathbb{Z}$.

(c) A one-to-one function $f : \mathbb{N} \to [0, 1]$.

(d) A one-to-one function $f : A \to B$ and an onto function $g : B \to C$ such that $g \circ f$ is not one-to-one.

(e) True or False: Let A and B be sets, and suppose $f : A \to B$ is one-to-one. Then there exists an onto function $g : B \to A$. Give a proof or counterexample.

Are the following functions bijective? Give a proof or explain why not.

(f) $f : P \to P$, where P is the set of people and $f(p)$ is defined to be p's mother.

(g) $f : (0, 1) \to (-2, 2)$ defined by $f(x) = 4x - 2$.

(h) $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined by $f(x) = (x, x)$.

4. **Cardinality (Ch. 5).** The most important thing to know from this chapter is the definition of $A \approx B$ (5.1) and that it is an equivalence relation. You should also know the definitions of Finite, Infinite, and Countably Infinite Sets. It might be helpful to know how to find bijections between different infinite sets such as subsets of $\mathbb{R}$, or subsets of $\mathbb{Z}$.

(a) Suppose $A \approx C$ and $B \approx D$. Prove that $A \times B \approx C \times D$.

(b) Show that $\mathbb{N} \approx \mathbb{N} \times \{0, 1\}$.