Math 8 - Practice Final Exam Solutions

Spring 2009

1. Write each of the following sets in set-builder notation $\{x \in U \mid P(x)\}$, where U is a set and $P(x)$ is a proposition depending on x.

   (a) $A = \{0, 4, 16, 36, 64, 100, \ldots\}$
   Solution. $A = \{x \in \mathbb{Z} \mid \exists n \in \mathbb{Z}\ x = (2n)^2\}$

   (b) $B = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, \ldots\}$
   Solution. $B = \{n \in \mathbb{N} \mid\ \sim (3|n)\}$

2. Are the following propositions True or False? Give brief justifications for your answers. (The domain of interpretation for all variables is $\mathbb{Z}$.)

   (a) $\forall x\ \exists y\ \exists z\ (x + y = 2z)$.
   Solution. TRUE. It says that for any x we can add some number y to get an even number $2z$. This is clearly satisfied by choosing $y = x$ and $z = x$.

   (b) $\forall x\ \exists y\ (xy = x + y)$.
   Solution. FALSE. It does not hold for $x = 1$, since $y = 1 + y$ is never true for an integer y.

3. Write the following statements using symbols only, and no words. (You do not have to prove them.)

   (a) "Every rational number is a real number."
   Solution. $\mathbb{Q} \subseteq \mathbb{R}$ or $\forall x \in \mathbb{Q}\ (x \in \mathbb{R})$

   (b) "The product of any two odd integers is odd."
   Solution. $\forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ [\sim (2|x) \wedge \sim (2|y)] \Rightarrow \sim (2|xy)$.

4. For two subsets A, B of a set U, let $A \bigodot B = \widetilde{A \cup B} = U - (A \cup B)$. Draw Venn diagrams illustrating the following subsets of U.

   (a) $A \bigodot B$ **Solution.** The picture should have everything outside of $A \cup B$ shaded in.

   (b) $\tilde{A} \bigodot \tilde{B}$ (note $\tilde{A} = U - A$ is the complement of A.)
   Solution. The picture should have just $A \cap B$ shaded in.

   (c) $(B - A) \bigodot (A - B)$.
   Solution. The picture should have $A \cap B$ and everything outside of $A \cup B$ shaded in.

5. Give examples of functions $f : A \to B$ and $g : B \to C$ with the stated properties, or briefly explain why none can exist. (Be sure to specify the sets A, B, C in your examples.)

(a) f is one-to-one, but $g \circ f$ is not one-to-one.

Solution. Let $A = \{1, 2\}$, $B = \{3, 4\}$ and $C = \{5\}$, and let $f = \{(1, 3), (2, 4)\}$ and $g = \{(3, 5), (4, 5)\}$. Then f is one-to-one since $f(1) \neq f(2)$, but $g(f(1)) = g(3) = 5 = g(4) = g(f(2))$ so $g \circ f$ is not injective.

(b) $g \circ f$ is one-to-one, but f is not one-to-one.

Solution. No example exists. Suppose $g \circ f$ is one-to-one, and let $a, b \in A$. If $f(a) = f(b)$, then $g(f(a)) = g(f(b))$. But $g \circ f$ is one-to-one, so we must have $a = b$. This shows that f must be one-to-one.

6. Consider the relation $\equiv$ on $\mathbb{R}$, defined by

$$a \equiv b \Leftrightarrow a - b \in \mathbb{Z}$$

for any $a, b \in \mathbb{R}$.

(a) Show that $\equiv$ defines an equivalence relation on $\mathbb{R}$.

Solution. Reflexive: For any $a \in \mathbb{R}$, we have $a \equiv a$ since $a - a = 0 \in \mathbb{Z}$.

Symmetric: Suppose $a \equiv b$. This means that $a - b \in \mathbb{Z}$. Thus $b - a = -(a - b) \in \mathbb{Z}$, and hence $b \equiv a$.

Transitive: Suppose $a \equiv b$ and $b \equiv c$. This means that $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$. Thus $a - c = (a - b) + (b - c) \in \mathbb{Z}$, and $a \equiv c$.

(b) Describe the equivalence classes of 1 and $\frac{1}{2}$.

Solution. $1/\equiv = \{x \in \mathbb{R} \mid x \equiv 1\} = \{x \in \mathbb{R} \mid x - 1 \in \mathbb{Z}\} = \mathbb{Z}$.

$$\begin{aligned} \frac{1}{2}/\equiv &= \{x \in \mathbb{R} \mid x \equiv \frac{1}{2}\} \\ &= \{x \in \mathbb{R} \mid x - \frac{1}{2} \in \mathbb{Z}\} \\ &= \{y + \frac{1}{2} \mid y \in \mathbb{Z}\} \\ &= \{\ldots, -1.5, -0.5, 0.5, 1.5, 2.5, \ldots\}. \end{aligned}$$

7. Let $A = \{0, 1, 2\}$, and let B be the set of all functions $f : A \to A$. Let $e : B \to A$ be the function defined by

$$e(f) = f(0) \quad \text{for any function } f : A \to A.$$

Is the function e one-to-one, onto, or bijective? Justify your answer. (You should explain why it **does** have any of these properties, and ALSO why it **does not** have the other properties)

Solution. e is not injective: Notice that $|B| = 3^3 = 27$, while $|A| = 3$. By the pigeonhole principle, no function from B to A can be injective. (Alternatively, if

$f = \{(0,0),(1,1),(2,2)\}$ and $g = \{(0,0),(1,0),(2,0)\}$, then $g \neq f$, but $e(f) = f(0) = 0 = g(0) = e(g)$.)

e is onto: Let $a \in A$, and let f be a function from A to A that sends 0 to a. For instance $f = \{(0,a),(1,0),(2,0)\}$. Then $e(f) = f(0) = a$.

e is not bijective: Since e is not injective, it cannot be bijective.

8. Suppose that A and B are sets with equal power sets, that is, $\mathcal{P}(A) = \mathcal{P}(B)$. Prove that $A = B$.

   Solution. Assume $\mathcal{P}(A) = \mathcal{P}(B)$. Since $A \subseteq A$, $A \in \mathcal{P}(A)$ by definition of the power set. Thus

$$\begin{aligned} A \subseteq A &\Rightarrow A \in \mathcal{P}(\mathcal{A}) \\ &\Rightarrow A \in \mathcal{P}(B) \\ &\Rightarrow A \subseteq B. \end{aligned}$$

   Similarly, we can swap the roles of A and B to get

$$\begin{aligned} B \subseteq B &\Rightarrow B \in \mathcal{P}(\mathcal{B}) \\ &\Rightarrow B \in \mathcal{P}(A) \\ &\Rightarrow B \subseteq A. \end{aligned}$$

   Since $A \subseteq B$ and $B \subseteq A$, we must have $A = B$.

9. Let x be a real number such that $x \geq 1$. Prove by induction that

$$x^n - 1 \geq (x-1)^n$$

   for any integer $n \geq 1$.

   Solution. Let $P(n)$ be the proposition $x^n - 1 \geq (x-1)^n$

   Basis step: Let $n = 1$. $P(1)$ says $x - 1 \geq x - 1$, which is obviously true.

   Inductive Step: Let $k \geq 1$ and assume $x^k - 1 \geq (x-1)^k$. We now prove $P(k+1)$.

$$\begin{aligned} (x-1)^{k+1} &= (x-1)^k(x-1) \\ &\leq (x^k-1)(x-1) \\ &= x^{k+1} - x^k - x + 1 \\ &\leq x^{k+1} - 1 - 1 + 1, \quad \text{Since } x \geq 1 \text{ and } x^k \geq 1, \\ &= x^{k+1} - 1 \end{aligned}$$

   By induction $P(n)$ is true for all integers $n \geq 1$.