Math 8, Midterm Review
Spring 2009

The problems on the midterm will be similar to the ones from the homework assignments, lectures, and this review sheet. As usual, you will be required to show your work and fully justify your answers to receive full credit. The test will focus on sections 1.1-1.6 and 2.1-2.3 from the text. Below is a summary of the topics and types of problems you should be able to do, along with some practice problems. Solutions to these will be available later.

- **Propositions and Logical Connectives (1.1-1.2)**
  - Know how to translate back and forth between English sentences and propositional forms involving the logical connectives: $\wedge, \vee, \sim, \Rightarrow, \Leftrightarrow$.
  - Know how to construct truth tables and check logical equivalence between two propositional forms (Theorem 1.2, p. 15 may be helpful, but parts (a), (b), (d), (f) are the only ones that you should have memorized).
  - Know how to find the converse and contrapositive of a conditional statement.

1. Make a truth table for the propositional form $(P \wedge \sim Q) \Rightarrow (P \vee Q)$, and then find a simpler expression which is logically equivalent to it.
2. Consider the implication "If it rains sometimes, then nobody is happy." Write English sentences for (a) the converse; (b) the contrapositive; and (c) the negation of this statement. (Your answers should be as simple and natural as possible, so you should avoid phrases like "It is not the case that..." or "It is false that...", etc.)

- **Quantifier Notation (1.3)**
  - Know the meanings and correct usage of $\exists$ = "There exists" and $\forall$ = "for all". You should be able to write propositions using this notation.
  - De Morgan's laws for negations of quantifiers: $\sim (\exists x P(x)) \equiv \forall x \sim P(x)$ and $\sim (\forall x P(x)) \equiv \exists x \sim P(x)$.
  - You should be able to decide whether propositions involving quantifiers are true or false, and give brief justifications.

3. True or False? Give brief justifications. (The universe of discourse is $\mathbb{R}$.)
   (a) $\forall x\ \forall y\ (xy \geq 0)$
   (b) $\exists a\ \forall b\ (ba = b/a)$
4. Let $P(x, y)$ stand for the proposition "x and y are friends", and assume the universe of discourse is the set of all people. Express the following propositions symbolically:
   (a) "Every two people have a common friend."
   (b) "Nobody is friends with everyone."

- **Proof Techniques (1.4-1.7)**
  - Know how to formulate Direct and Indirect Proofs, as well as Proofs by Contradiction.
  - Knowing the definitions of terms like even, odd, rational, irrational, divides, prime.

5. In this problem x and y are real numbers. Consider the proposition: "If the product xy is irrational, then either x or y is irrational."
   (a) State the contrapositive of this proposition.
   (b) Prove this proposition.
   (c) Is the converse of this proposition also true (for all real numbers x, y)? Explain.

- **Sets: Definitions and Notation (2.1)**
  - Know the meanings and correct usage of the symbols $\in, \notin, \{\cdots \mid \cdots\}, \emptyset$: For example, you should be able to express propositions in this notation.
  - Know how to read and write both variations of Set-Builder notation that we discussed:
    1) $\{x \in S \mid P(x)\ \}$ defines the set of exactly those elements of S that make $P(x)$ true.
    2) $\{f(x) \mid x \in S\ \}$ defines the set of all elements of the form $f(x)$ where x is any element of S.
  - Set Equality. $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$. You should be able to prove that two sets are equal by using this definition.

6. Write the following sets in the form a) $\{x \in S \mid P(x)\ \}$ and the form b) $\{f(x) \mid x \in S\ \}$.
   (i) $\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots\}$
   (ii) $\{11, 21, 31, 41, \ldots\}$
7. List (or otherwise describe) the elements of the set $\{5x - 1 \mid x \in \mathbb{Z}\}$.
8. Prove that $\{5x - 1 \mid x \in \mathbb{Q}\} = \mathbb{Q}$.

- **Set Inclusion (2.1)**
  - Know the meaning and usage of the symbols $\subseteq$, $\subset$. You should be able to express propositions symbolically using this notation.

- Know the definitions of subsets and proper subsets:
  1) A is a subset of $B \Leftrightarrow A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$.
  2) A is a proper subset of $B \Leftrightarrow A \subset B \Leftrightarrow (A \subseteq B) \wedge (A \neq B)$.
  You should be able to prove, using these definitions that one set is a subset of another.
- You should be able to prove set equality by using the identity

$$A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A).$$

- **Set Operations: Union, Intersection, Complement (2.2)**
  - Know the definitions of $A \cup B$, $A \cap B$, $B - A$, $\tilde{A}$. Be able to illustrate sets obtained by combinations of these operations on sets A, B, C using a Venn diagram.
  - Theorem 2.6 (p. 80) may be helpful to know, but more important is that you should be able to prove these statements using the definitions.
  - Theorem 2.7 (p. 82) is also helpful, especially if you need to show that two sets are equal by showing that each one is a subset of the other.

9. Shade in the region corresponding to $S = A \cap (B \cup C)$ on a Venn diagram. On a separate diagram, shade in the region corresponding to $T = A \cup (B \cap C)$. If these are different, give an example of sets A, B, C and an element x that belongs to one of the sets S, T but not the other.
10. Let A and B be sets. Prove that $A \cap B = A \cup B$ if and only if $A = B$.

- **Indexed Families of Sets (2.3)**
  - Understand the definition: If I is a set, a **family of sets indexed by** I, is a collection of sets A_i, one for each element $i \in I$. This may also be written $\{A_i \mid i \in I\}$ or $\{A_i\}_{i \in I}$.
  - Know the definitions (p. 86-87) for the union $\bigcup_{i \in I} A_i$ and intersection $\bigcap_{i \in I} A_i$ of an indexed family of sets $\{A_i\}_{i \in I}$.

11. For each $i \in \mathbb{N}$, let $A_i = \{x \in \mathbb{N} \mid x \geq i\}$. Compute $\bigcup_{i \in \mathbb{N}} A_i$ and $\bigcap_{i \in \mathbb{N}} A_i$. (Give brief justifications for your answers, but not rigorous proofs.)

- **The Power Set (2.1)**
  - Know the definition of the power set of a set: For a set A, the power set of A is the set $\mathcal{P}(A) = \{X \mid X \subseteq A\}$. This is the set of all subsets of A. This can also be expressed

$$X \subseteq A \Leftrightarrow X \in \mathcal{P}(A).$$

  You should be able to compute the power sets of some small sets.

12. What is $\mathcal{P}(\{a, \{b, c\}\})$?

13.* Is it always true that $A \cap \mathcal{P}(A) = \emptyset$? Provide a justification or a counterexample.