Solutions to Midterm Review Problems
Math 8, Spring 2009

- **Propositions and Logical Connectives (1.1-1.2)**

1. Make a truth table for the propositional form $(P\wedge \sim Q) \Rightarrow (P \vee Q)$, and then find a simpler expression which is logically equivalent to it.

   Solution. The truth table should have a T in the rows where Q is True and an F in the rows where Q is False.

$$\begin{aligned}(P\wedge \sim Q) \Rightarrow (P \vee Q) &\equiv \sim (P\wedge \sim Q) \vee (P \vee Q)\\ &\equiv (\sim P \vee Q) \vee (P \vee Q)\\ &\equiv \sim P \vee P \vee Q \vee Q\\ &\equiv Q.\end{aligned}$$

2. Consider the implication "If it rains sometimes, then nobody is happy." Write English sentences for (a) the converse; (b) the contrapositive; and (c) the negation of this statement. (Your answers should be as simple and natural as possible, so you should avoid phrases like "It is not the case that..." or "It is false that...", etc.)

   Solution. (a) "If nobody is happy, then it rains sometimes."

   (b) "If somebody is happy, then it never rains."

   (c) "It rains sometimes and somebody is happy."

- **Quantifier Notation (1.3)**

3. True or False? Give brief justifications. (The universe of discourse is $\mathbb{R}$.)

   (a) $\forall x\ \forall y\ (xy \geq 0)$

   (b) $\exists a\ \forall b\ (ba = b/a)$

   Solution. (a) This says that the product of any two real numbers x and y is nonnegative. This is FALSE! For instance, $-1 * 1 = -1 < 0$. In fact, any two numbers x and y whose product is negative (eg. one is positive and the other is negative) provide a counterexample.

   (b) This says that there is a real number a such that $ba = b/a$ is always true. This is TRUE! Simply let $a = 1$, as $b * 1 = b/1$ for all $b \in \mathbb{R}$. $a = -1$ would also work.

4. Let $P(x, y)$ stand for the proposition "x and y are friends", and assume the universe of discourse is the set of all people. Express the following propositions symbolically:

   (a) "Every two people have a common friend."

   (b) "Nobody is friends with everyone."

Solution. (a) $\forall x\ \forall y\ \exists z\ (P(x,z) \wedge P(y,z))$

(b) $\sim \exists x\ \forall y\ P(x,y)$. This is also equivalent to $\forall x\ \exists y\ \sim P(x,y)$, which corresponds to the equivalent (but somewhat ambiguous) statement "Everyone is not friends with someone."

- **Proof Techniques (1.4-1.7)**

5. In this problem x and y are real numbers. Consider the proposition: "If the product xy is irrational, then either x or y is irrational."
   (a) State the contrapositive of this proposition.
   (b) Prove this proposition.
   (c) Is the converse of this proposition also true (for all real numbers x, y)? Explain.

   Solution. (a) The conclusion can be written $\sim (x \in \mathbb{Q}) \vee \sim (y \in \mathbb{Q})$. By De Morgan's law, this is equivalent to $\sim (x \in \mathbb{Q} \wedge y \in \mathbb{Q})$. Thus the negation of the conclusion is "x and y are rational." The negation of the hypothesis is "xy is rational." Thus the contrapositive says "If x and y are rational, then xy is rational."

   (b) We prove it indirectly by proving the contrapositive stated in (a). Assume x and y are rational. Then $x = a/b$ and $y = c/d$ for integers a, b, c, d with $b, d \neq 0$. Thus $xy = ac/bd$ is rational, since ac and bd are both integers and $bd \neq 0$.

   (c) The converse says that "If x or y is irrational, then xy is irrational." This is obviously false, as demonstrated by the easy example $0 \cdot \sqrt{2} = 0$, which works since $x = xy = 0$ is rational, but $y = \sqrt{2}$ is not.

- **Sets: Definitions and Notation (2.1)**

6. Write the following sets in the form a) $\{x \in S \mid P(x)\ \}$ and the form b) $\{f(x) \mid x \in S\ \}$.
   (i) $\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots\}$
   (ii) $\{11, 21, 31, 41, \ldots\}$
   Solution. (i) a) $\{x \in \mathbb{R} \mid x^2 \in \mathbb{N} \wedge x > 0\}$; b) $\{\sqrt{x} \mid x \in \mathbb{N}\}$
   (ii) a) $\{x \in \mathbb{N} \mid 10|(x-1)\}$; b) $\{10x + 1 \mid x \in \mathbb{N}\}$

7. List (or otherwise describe) the elements of the set $\{5x - 1 \mid x \in \mathbb{Z}\}$.
   Solution. The elements of this set are all the integers that are one less than a multiple of 5; i.e., it equals the set $\{\ldots, -6, -1, 4, 9, \ldots\}$.

8. Prove that $\{5x - 1 \mid x \in \mathbb{Q}\} = \mathbb{Q}$.
   Solution. We will show that $\{5x-1 \mid x \in \mathbb{Q}\} \subseteq \mathbb{Q}$ and that $\mathbb{Q} \subseteq \{5x-1 \mid x \in \mathbb{Q}\}$. (This is all we need to show since we know that for any two sets A and B: $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$.)

$\{5x-1 \mid x \in \mathbb{Q}\} \subseteq \mathbb{Q}$: Let $y \in \{5x-1 \mid x \in \mathbb{Q}\}$. This means that $y = 5x-1$ for some rational number x. Clearly such a y must also be rational (if $x = a/b$ for $a, b \in \mathbb{Z}$, then $y = 5x - 1 = (5a - b)/b$). Hence $y \in \mathbb{Q}$.

$\mathbb{Q} \subseteq \{5x-1 \mid x \in \mathbb{Q}\}$: Let $y \in \mathbb{Q}$. We want to show that $y = 5x-1$ for some $x \in \mathbb{Q}$. If this were true, we can see that $x = (y+1)/5$. So let $x = (y+1)/5$ and note that $x \in \mathbb{Q}$ (if $y = a/b$ for $a, b \in \mathbb{Z}$, then $x = (y+1)/5 = (a+b)/5b$). Hence $y = 5x - 1$ with $x = (y+1)/5 \in \mathbb{Q}$, showing that y is an element of the second set too.

- **Set Operations: Union, Intersection, Complement (2.2)**

9. Shade in the region corresponding to $S = A \cap (B \cup C)$ on a Venn diagram. On a separate diagram, shade in the region corresponding to $T = A \cup (B \cap C)$. If these are different, give an example of sets A, B, C and an element x that belongs to one of the sets S, T but not the other.

   Solution. No pictures here, but the shaded regions are definitely different. The first only contains that part of A that is overlapped by either B or C, while the second contains all of A. For the example, let $A = \{1,2\}, B = \{2,3\}$ and $C = \{2,4\}$. Then $S = \{2\}$, while $T = \{1,2\}$, so $S \neq T$.

10. Let A and B be sets. Prove that $A \cap B = A \cup B$ if and only if $A = B$.

    Solution. $\Rightarrow$: Assume that $A \cap B = A \cup B$. We will show that $A \subseteq B$ and $B \subseteq A$, since this is equivalent to $A = B$. We have the following subset inclusions

    $$A \subseteq A \cup B = A \cap B \subseteq B$$

    and

    $$B \subseteq A \cup B = A \cap B \subseteq A,$$

    from which we obtain $A \subseteq B$ and $B \subseteq A$.

    $\Leftarrow$: Assume $A = B$. Then $A \cap B = A \cap A = A$ and $A \cup B = A \cup A = A$. So $A \cap B = A = A \cup B$.

- **Indexed Families of Sets (2.3)**

11. For each $i \in \mathbb{N}$, let $A_i = \{x \in \mathbb{N} \mid x \geq i\}$. Compute $\bigcup_{i \in \mathbb{N}} A_i$ and $\bigcap_{i \in \mathbb{N}} A_i$. (Give brief justifications for your answers, but not rigorous proofs.)

    Solution. We begin by writing down the sets A_i for a few values of i.

    $A_1 = \{x \in \mathbb{N} \mid x \geq 1\} = \{1, 2, 3, \ldots\}$

    $A_2 = \{x \in \mathbb{N} \mid x \geq 2\} = \{2, 3, 4, \ldots\}$

    $A_3 = \{x \in \mathbb{N} \mid x \geq 3\} = \{3, 4, 5, \ldots\}$

    $\bigcup_{i \in \mathbb{N}} A_i$ is the set of all numbers that belong to at least one A_i. Since all natural numbers already belong to A_1, this union will be $\mathbb{N}$.

$\bigcap_{i\in\mathbb{N}} A_i$ is the set of all numbers that belong to all of the A_i's. But as i grows, each set A_i excludes more numbers (namely, A_i excludes all the numbers less than i), and we can see that no natural number will be contained in *every* A_i. For instance, $n \notin A_{n+1} = \{n+1, n+2, n+3, \ldots\}$. Thus, the intersection is the empty set $\emptyset$.

- **The Power Set (2.1)**

12. What is $\mathcal{P}(\{a, \{b, c\}\})$?
    Solution. $\mathcal{P}(\{a, \{b, c\}\}) = \{\emptyset, \{a\}, \{\{b, c\}\}, \{a, \{b, c\}\}\}$

13.* Is it always true that $A \cap \mathcal{P}(A) = \emptyset$? Provide a justification or a counterexample.
    Solution. No! The simplest way to find an example of a set A such that $A \cap \mathcal{P}(A) \neq \emptyset$ is to notice that $\mathcal{P}(A)$ always contains $\emptyset$ as an element. Thus if A is any set containing the empty set $\emptyset$ as an element, $\emptyset$ will also be an element of the intersection $A \cap \mathcal{P}(A)$. Since this intersection has an element, it is nonempty.

    For example, if $A = \{\emptyset\}$, then $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}\}$, and $A \cap \mathcal{P}(A) = \{\emptyset\}$. Another example can be obtained by letting $A = \{a, \{a\}\}$, since then $\{a\}$ is also an element of $\mathcal{P}(A)$.