Math 8 - Homework #4 Solutions

1.4: 7. (i) Assume that a and b are positive integers and that $a|b$ and $b|a$. Then $b/a \in \mathbb{N}$ and $a/b \in \mathbb{N}$ and $(b/a)(a/b) = 1$. The only two natural numbers whose product is 1 are 1 and 1. Thus $b/a = a/b = 1$ and it follows that $a = b$.

(j) Assume $a|b$ and $c|d$. Thus $b/a, d/c \in \mathbb{Z}$, and hence their product bd/ac is also an integer. Hence, by definition, $ac|bd$.

1.4: 8. Let n be a natural number.

Case 1: Assume n is odd. Then $n = 2k + 1$ for an integer k. Thus

$$n^2 + n + 3 = (2k + 1)^2 + (2k + 1) + 3 = 4k^2 + 6k + 5 = 2(2k^2 + 3k + 2) + 1$$

is odd.

Case 2: Assume n is even. Then $n = 2k$ for some integer k. Thus

$$n^2 + n + 3 = (2k)^2 + (2k) + 3 = 4k^2 + 2k + 3 = 2(2k^2 + k + 1) + 1$$

is odd.

1.4: 9. (a) Working backwards: If $\frac{x+y}{2} \geq \sqrt{xy}$, then we can square both sides to get $(x + y)^2/4 \geq xy$. Multiplying by 4 and then subtracting $4xy$ from both sides gives $(x-y)^2 \geq 0$. Since we know that this inequality is true (the square of any real number is always non-negative), we can formulate a proof that starts from this known inequality by reversing the above steps.

Proof. We know that $(x - y)^2 \geq 0$. Adding $4xy$ to both sides yields $(x + y)^2 \geq 4xy$. Now, divide both sides by 4 to get $(x + y)^2/4 \geq xy$. Finally, since $x, y > 0$ (by assumption), the inequality is preserved when we take the square root of both sides. Thus we have $\frac{x+y}{2} \geq \sqrt{xy}$.

1.5: 4. (a) The contrapositive of the statement is "If $x \geq 0$, then $x^2 + 2x \geq 0$." Thus we assume that $x \geq 0$. Then $x + 2 \geq 0$ as well. Hence $x^2 + 2x = x(x + 2) \geq 0$.

1.5: 6. (a) Assume that $a, b \in \mathbb{N}$ such that $a|b$ and $a > b$. Then $b/a \in \mathbb{N}$ since a divides b, but $a > b$ implies that $b/a < 1$. Since 1 is the smallest natural number, we have $b/a \geq 1$, which contradicts $b/a < 1$. Therefore, if $a|b$ then $a \leq b$.

(d) Assume that $a, b \in \mathbb{N}$ such that $a - b$ is odd and $a + b$ is not odd. Then $a + b$ is even, and $2b = (a + b) - (a - b)$ must be odd, since it is an even number minus an odd number. But then, $2b$ would be simultaneously even and odd, which is impossible. Therefore, if $a - b$ is odd, then so is $a + b$.

1.5: 7. (b) $\Rightarrow$: Assume first that $(a + 1)|b$ and $b|(b + 3)$. Then we can write $b = n(a + 1)$ and $b + 3 = mb$ for some positive integers n and m. It follows that $3 = (m - 1)b$, and since 3 is prime and b is (assumed) positive, we must have either $b = 1$ or $b = 3$. If

$b = 1$, then $(a+1)|b$ implies $a + 1 \leq b = 1$, which implies $a \leq 0$. However, this would contradict the assumption that a is positive. Thus, we can only have $b = 3$. Then $(a+1)|b$ implies that $a + 1 = 1$ or $a + 1 = 3$. Since $a > 0$, we can only have $a = 2$.

$\Leftarrow$: Conversely, assume that $a = 2$ and $b = 3$. Then $a+1 = 3 = b$ so we have $(a+1)|b$. Since $b = 3$ and $b + 3 = 6$ we also have $b|(b+3)$.

1.5: 12. (a) F. You are asked to prove $P \Rightarrow Q$ (P = "m^2 is odd", etc.), but the proof verifies $\sim P \Rightarrow \sim Q$, which is the contrapositive of the converse of $P \Rightarrow Q$, and thus not equivalent to the proposition to be approved. While the logic in the proof appears correct, the truth of this statement has no bearing on the truth of the proposition that needs to be proved.

(b) A. This is a perfect example of an indirect proof (i.e., proof by contraposition).

(c) A. This is a perfect example of a proof by contradiction.

(d) C. (or A-) The argument is basically correct. However, in the third sentence it should be stated that we ASSUME $a|b$ and we ASSUME $a|c$, since this information is not given.

1.6: 4. Assume that p is a prime number and $p \neq 3$. Since p is not a multiple of 3, when we divide p by 3 the remainder will be either 1 or 2. We consider these two cases separately.

Case 1: $p = 3k + 1$ for some integer k. $p^2 + 2 = (3k+1)^2 + 2 = 9k^2 + 6k + 3 = 3(3k^2 + 2k + 1)$ is divisible by 3.

Case 2: $p = 3k + 2$ for some integer k. $p^2 + 2 = (3k+2)^2 + 2 = 9k^2 + 12k + 6 = 3(3k^2 + 4k + 2)$ is divisible by 3.

In either case, we have $3|(p^2 + 2)$.