Math 8 - Homework #5

Due: May 5, 2009

1. Express each of the following statements using sets. Your answers should be of the form "[something] $\in$ (or $\notin$) [some set]".

   (a) x is a nonnegative integer that is smaller than 5.

   (b) Either a or b equals 1.

   (c) Neither x nor y is 0.

   Solution. (a) $x \in \{1, 2, 3, 4\}$

   (b) $1 \in \{a, b\}$

   (c) $0 \notin \{x, y\}$

2. Write each of the sets below in two ways: a) in the form $\{x \in U \mid P(x)\}$, and b) in the form $\{f(x) \mid x \in S\}$ where $f(x)$ is a function (possibly of multiple variables), and S and U are some sets.

   (i) $A = \{1, 2, 4, 8, 16, \ldots\}$ is the set of all (integer) powers of 2.

   (ii) B is the set of all integers that can be written as the sum of two perfect squares.

   (iii) C is the set of all the reciprocals of natural numbers.

   Solution. (i) $A = \{x \in \mathbb{N} \mid \exists n \in \mathbb{Z}\ (x = 2^n)\} = \{2^n \mid n \in \mathbb{N} \cup \{0\}\}$.

   (ii) $B = \{n \in \mathbb{Z} \mid \exists x, y \in \mathbb{Z}(n = x^2 + y^2)\} = \{x^2 + y^2 \mid x, y \in \mathbb{Z}\}$.

   (iii) $C = \{x \in \mathbb{R} \mid \frac{1}{x} \in \mathbb{N}\} = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

3. (a) Prove that $\{2k - 1 \mid k \in \mathbb{Z}\} = \{2k + 1 \mid k \in \mathbb{Z}\}$.

   Solution. For convenience, let $A = \{2k - 1 \mid k \in \mathbb{Z}\}$ and $B = \{2k + 1 \mid k \in \mathbb{Z}\}$. We must show that for any x, $x \in A \Leftrightarrow x \in B$.

   $x \in A \Rightarrow x \in B$: Assume that $x \in A$. By the definition of A, this means that $x = 2k - 1$ for some $k \in \mathbb{Z}$. Thus $x = 2k - 1 = 2k - 2 + 1 = 2(k - 1) + 1$. Since $k - 1$ is an integer, x is also an element of B.

   $x \in A \Leftarrow x \in B$: Now assume that $x \in B$. By definition, $x = 2k + 1$ for some integer k. Thus $x = 2k + 1 = 2k + 2 - 1 = 2(k + 1) - 1$. Since $k + 1$ is an integer, x is also an element of A.

   (b) Are the sets $\{2k - 1 \mid k \in \mathbb{N}\}$ and $\{2k + 1 \mid k \in \mathbb{N}\}$ also equal? Justify your answer. (Suggestion: start listing the elements in these sets by plugging in different natural numbers for k.)

   Solution. No, these sets are not equal. The first is $\{1, 3, 5, 7, \ldots\}$, but the second is $\{3, 5, 7, \ldots\}$. The first contains 1, but the second does not.

4. **Exercises 2.1 p. 76-77:** 5)b; 13; 19)a–d.

   Solutions. 5. (b): Any example where A, B, C are all equal to each other will work. (In fact, this is the only possibility since $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$, and $A \subseteq C$ together with $C \subseteq A$ implies $A = C$, and then $B = A$ will follow too.)

   13. (a)-(c): All True. By definition of the set X, we have $a \in X \Leftrightarrow P(a)$. Parts (a) and (b) state the two directions of this biconditional, while (c) states the contrapositive of (a).

   19. (a): C. The proof shows only $Y \subseteq X$, and does not address the other inclusion $X \subseteq Y$, which is needed to conclude that $X = Y$.

   (b); F. Only one example is given. A valid proof should assume nothing about the sets A, B, C beyond the given hypotheses: $A \subseteq B$ and $B \subseteq C$.

   (c): C (or maybe A?). The statement "Thus, $x \in C$." appears to be about any object x. Obviously, not every object is necessarily an element of C. Instead, it should say "Thus, if $x \in A$, then $x \in C$."

   (d): F. The definition of $\subseteq$ is backwards.

5. **Exercises 2.2 p. 83-84:** 2)d, f; 10)f (it may help to draw a Venn diagram); 12)b,c (you may draw a Venn diagram);

   Solutions. 2. (d) $P - N$ is the set P of all positive integers.

   (f) $\tilde{N}$ is the set of all non-negative integers.

   10. (f) Suppose that $A \subseteq C$ and $B \subseteq C$. To show $A \cup B \subseteq C$, we must show that for all $x \in A \cup B$, we have $x \in C$. Assume $x \in A \cup B$. Then, either $x \in A$ or $x \in B$. If $x \in A$, then $A \subseteq C$ implies $x \in C$. Similarly, if $x \in B$, then $B \subseteq C$ implies $x \in C$. Thus, in either case, $x \in C$ as required.

   12. (b) Let $A = \{1\} = C$ and $B = \{1, 2\}$.

   (c) Let $A = \{1\}$, $B = \{2\}$ and $C = \{1, 2\}$.

6. **Exercises 2.3 p. 92-93:** 1)d, j, m.

   Solutions. 1. (d) $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$ and $\bigcup_{n\in\mathbb{N}} B_n = \mathbb{N} - \{1\}$.

   (j) $\bigcap_{n\in\mathbb{N}} M_n = \{0\}$ since M_n consists of all integer multiples of n and 0 is the only integer that is a multiple of all integers. $\bigcup_{n\in\mathbb{N}} M_n = \mathbb{N}$ since the union contains $M_1 = \mathbb{N}$ as a subset.

   (m) $\bigcap_{n\in\mathbb{Z}} A_n = \emptyset$ since even any two sets A_n and A_m with $n \neq m$ have no elements in common. $\bigcup_{n\in\mathbb{Z}} A_n = \mathbb{R} - \mathbb{Z}$ the set of all real numbers that are not integers, since the set A_n consists of all the real numbers x with $n < x < n + 1$.