Name: Solutions

Perm No.: ________________

Section Time (circle one): 4 pm 5pm

Math 8 - Midterm

May 6, 2009

Instructions:

- This exam consists of 4 problems worth 10 points each, for a total of 40 points.
- You must show all your work and fully justify your answers in order to receive full credit. Partial credit will be given for work that is relevant and correct. Your proofs will be graded for clarity and organization, in addition to correctness.
- No books, notes or calculators are allowed.
- Write your answers on the test itself, in the space alotted. You may attach additional pages if necessary.

1	
2	
3	
4	
Total	

1. (a) Make a truth table for the propositional form R below.

$$R: \quad (Q \Rightarrow \sim P) \Rightarrow (P \wedge \sim Q)$$

(You should show your work by including some intermediary columns in your truth table. If you wish, you may do (b) first.)

P	Q	$Q \Rightarrow \sim P$	$P \wedge \sim Q$	R
T	T	F	F	T
T	F	T	T	T
F	T	T	F	F
F	F	T	F	F

(b) Use the truth table from (a) or known logical equivalences to find a simpler propositional form that is logically equivalent to R. (Can you find a simpler form that uses only 1 logical connective?)

$R \equiv P$ since the columns of R & P in the truth table are the same.

2. State whether the following propositions are True or False, and give a brief justification for your answer (a single example may suffice).

(a) $\forall a \in \mathbb{R}\ \exists b \in \mathbb{Z}\ (ab \leq 10)$.

$\boxed{\text{TRUE.}}$ for any $a \in \mathbb{R}$, choose $b = 0$

$ab = 0 \leq 10$.

(b) $\exists y \in \mathbb{Z}\ \forall x \in \mathbb{R}\ [(y > 0) \Rightarrow (x = y)]$.

$\boxed{\text{TRUE.}}$ choose y to be anything ≤ 0.

then $y > 0$ is False

and hence $(y > 0) \Rightarrow (x = y)$

is True for any x.

(c) The two sets are equal: $\{x \in \mathbb{R} \mid x^2 \in \mathbb{Z}\} = \{x^2 \in \mathbb{Z} \mid x \in \mathbb{R}\}$.

$\boxed{\text{FALSE.}}$

$$\{0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \ldots\} \neq \{0, 1, 2, 3, \ldots\}$$

3. For each $n \in \mathbb{Z}$ define the set $A_n = \{x \in \mathbb{R} \mid nx \in \mathbb{Z}\}$.

(a) Describe each of the the sets A_0, A_1 and A_2. (You may describe in words what the elements are, or list the elements if that is more convenient.)

$$A_0 = \{x \in \mathbb{R} \mid 0x \in \mathbb{Z}\} = \boxed{\mathbb{R}.}$$

$$\underbrace{0x = 0}_{\in \mathbb{Z}} \quad \forall x \in \mathbb{R}.$$

$$A_1 = \{x \in \mathbb{R} \mid x \in \mathbb{Z}\} = \boxed{\mathbb{Z}.}$$

$$A_2 = \{\underset{\frac{n}{2}}{x} \in \mathbb{R} \mid \underbrace{2x}_{n} \in \mathbb{Z}\} = \{0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \dots\}$$

$$\frac{n}{2} \Longleftarrow n$$

$$= \boxed{\{\tfrac{n}{2} \mid n \in \mathbb{Z}\}}$$

(b) Determine the set $\bigcup_{n\in\mathbb{N}} A_n$, and explain your reasoning. (Remember $0 \notin \mathbb{N}$).

$$A_n = \{x \in \mathbb{R} \mid \underbrace{nx}_{y} \in \mathbb{Z}\} = \{\tfrac{y}{n} \mid y \in \mathbb{Z}\}$$

= all rational numbers that can be written with n in denominator.

Since any rational number x can be written as a/b $a \in \mathbb{Z}, b \in \mathbb{N}$.

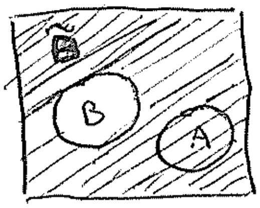
$x \in A_b$. Thus $x \in \bigcup_{n\in\mathbb{N}} A_n$

so $\boxed{\bigcup_{n\in\mathbb{N}} A_n = \mathbb{Q}}$

4. Let A and B be sets (contained in a universe U). Prove that

$$A \subseteq \tilde{B} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

(You may use Venn diagrams to help illustrate your argument, but you need to use the definitions of $\tilde{\ }$, $-$, $\cup$, etc.–or the basic identities relating them–to explain the pictures.)



$\Rightarrow$ Assume $A \subseteq \tilde{B}$

Thus $\forall x \ (x \in A \Rightarrow x \notin B)$.

Assume – by way of contradiction – that $A \cap B \neq \emptyset$. Let $x \in A \cap B$.

Then $x \in A$ and $x \in B$.

But $A \subseteq \tilde{B} \wedge x \in A \Rightarrow x \notin B$.

This contradiction shows $A \cap B = \emptyset$.

$\Leftarrow$ Assume $A \cap B = \emptyset$.

Let $x \in A$. We must show $x \notin B$.

By contradiction, assume $x \in B$

then $x \in A \wedge x \in B$. So $x \in A \cap B = \emptyset$

which is a contradiction

Thus $x \notin B$. & $A \subseteq \tilde{B}$.