1. For each of the following, state whether it is a Proposition, an Open Proposition, a Set, or none of the above. The letters $P, Q, R, \ldots$ represent Propositions, and the letters $A, B, C, \ldots$ represent Sets.

(a) $x \leq y$. Open Proposition. It is ambiguous as expressed, but will be either a True or False statement once values for $x$ and $y$ are specified.

(b) $\forall x \in \mathbb{R} \exists y \in \mathbb{N} (x \leq y)$. Proposition. It is equivalent to the True statement “For any real number, there is a larger natural number.” Any True statement is automatically a Proposition, and likewise for False statements.

(c) $\{x \in \mathbb{R} | x \leq 2\}$. Set. This is the set of all reals that are less than or equal to 2.

(d) $\{x \leq 2\}$. None of the Above. $x \leq 2$ without the curly braces would be an Open Proposition. But the curly braces suggest that we are defining a set. This could be referring to the same set as in (d), but this is poor notation, as it does not specify what Universe $x$ belongs to. For instance, would this be the set $\{x \in \mathbb{Z} | x \leq 2\}$ or would it be $\{x \in \mathbb{R} | x \leq 2\}$?

(e) $3 \in \{0, 1, 2\}$. Proposition. It is a False statement, and so automatically a proposition.

(f) $(P \iff Q) \subseteq \mathbb{R}$. None of the Above. This is nonsense: $P \iff Q$ denotes a biconditional Proposition, and a Proposition cannot be a subset of the Set $\mathbb{R}$. Compare this with the following Proposition, which makes sense since we are saying that one set is a subset of another set: $\{x \mid P(x) \iff Q(x)\} \subseteq \mathbb{R}$.

(g) $\mathbb{N} \subseteq \mathbb{R}$. Proposition. This is the True statement “The natural numbers are a subset of the real numbers.”

(h) $A = B \iff A \subseteq B$. Open Proposition. It is making a statement about two sets $A$ and $B$. Since $A$ and $B$ are not specified, it is Open.

(i) $\mathbb{Q} \cup \{x \in \mathbb{R} | x\sqrt{2} \in \mathbb{Q}\}$. Set. We have the union of two sets, so it is a set.

(j) $A \cap B \neq \emptyset$. Open Proposition. It is stating that $A$ and $B$ have a nonempty intersection, i.e., that they share at least one element. Since $A$ and $B$ are not specified, it is Open.

(k) $\{x \in \mathbb{R} | x \in \mathbb{Z}\} \iff \{x \in \mathbb{R} | -x \in \mathbb{Z}\}$. None of the Above. We have two sets with a Biconditional sign ($\iff$) between them. In general, $\iff$ can be place between 2 Propositions, or an Equality sign ($=$) can be placed between two Sets, to get a valid Proposition.
2. Below is a proof of the statement: For any two sets $A$ and $B$,

$$(A \subseteq B) \iff (A \subseteq A \cap B).$$

Describe what is wrong with the proof. Are any elements of it correct? How could you rewrite it so that it makes sense?

Proof.

$$A \subseteq B \; \overset{\text{def}}{=} \; \{ x \mid x \in A \Rightarrow x \in B \}$$

$$= \{ x \mid x \in A \Rightarrow (x \in B \land x \in A) \}$$

$$= \{ x \mid x \in A \Rightarrow x \in A \cap B \}$$

$$= A \subseteq A \cap B$$

Therefore $A \subseteq B \iff (A \subseteq A \cap B)$. Q.E.D.

The first mistake is in the first line: “$A \subseteq B$” is a (Open) Proposition, and so it makes no sense to say that it Equals the Set on the right hand side. The same error occurs on the last line, which states that a Set is equal to the Proposition “$A \subseteq A \cap B$”.

The equalities of Sets on the Right Side of the first 3 lines are essentially correct. This is because the propositions used to define these sets are logically equivalent. $x \in A \Rightarrow x \in B \equiv (x \in A \Rightarrow (x \in B \land x \in A))$ is a tautology, and this is equivalent to $x \in A \Rightarrow x \in A \cap B$ by the definition of the Intersection $A \cap B$.

To correct it, we can forget about the set notation (curly braces), and just use the definitions and tautologies to obtain a sequence of equivalent Propositions:

$$A \subseteq B \; \overset{\text{def}}{=} \; \forall x \; (x \in A \Rightarrow x \in B) \quad \text{(By definition of \; \subseteq .)}$$

$$\equiv \forall x \; (x \in A \Rightarrow (x \in B \land x \in A)) \quad \text{(By the tautology \; P \Rightarrow Q \equiv P \Rightarrow (P \land Q).)}$$

$$\equiv \forall x \; (x \in A \Rightarrow x \in A \cap B) \quad \text{(By definition of \; \cap .)}$$

$$\equiv A \subseteq A \cap B \quad \text{(By definition of \; \subseteq .)}$$

Therefore $A \subseteq B \iff (A \subseteq A \cap B)$. Q.E.D.