Atoms of a Countably Generated σ-Algebra

Daniel Epelbaum
Transcribed by Ashwin Trisal

Lemma 1 Every countably generated σ-algebra which is a collection of subsets of some set X contains an **atom**, an indecomposable nonempty element of the σ-algebra.

Let $(X, \mathcal{A})$ be such a σ-algebra, with generating set $\{A_i\}_{i\in\mathbb{N}}$. Define the equivalence relation

$$x \sim y \quad \iff \quad \left[\forall A \in \mathcal{A},\, x \in A \iff y \in A\right]$$

Then $X/\sim$ has an induced σ-algebra structure from $\mathcal{A}$. In fact, the σ-algebra generated by $\pi(\mathcal{A})$, where $\pi : X \to X/\sim$ is the quotient map, is just $\pi(\mathcal{A})$.
One direction is clear, because $\pi(\cup_{\alpha\in I} U_\alpha) = \cup_{\alpha\in I}\pi(U_\alpha)$ for any collection of sets $\{U_\alpha\}_{\alpha\in I}$. In particular, the closure under countable union of $\mathcal{A}$ implies the closure under countable union of $\pi(\mathcal{A})$.
Secondly, $\pi(\mathcal{A})$ is closed under countable intersection. In general, $\pi(\cap_\alpha U_\alpha) \subseteq \cap_\alpha \pi(U_\alpha)$. Moreover, in this specific case, if $[x] \in \cap_\alpha\pi(U_\alpha)$, then for each U_α, $x \in U_\alpha$. So $x \in \cap_\alpha U_\alpha$, so $[x] \in \pi(\cap_\alpha U_\alpha)$. So $\pi(\cap_\alpha U_\alpha) = \cap_\alpha\pi(U_\alpha)$, and $\mathcal{A}$ being closed under countable intersection implies that $\pi(\mathcal{A})$ is as well.
Finally, π being surjective implies that it commutes with the complement. So $\pi(\mathcal{A})$ is a σ-algebra.

Importantly, points in $\pi(\mathcal{A})$ correspond to atoms of $\mathcal{A}$. We show that $\pi(\mathcal{A})$ contains at least one point.

Lemma 2 If $(Y, \mathcal{B})$ is a σ-algebra generated by a collection of sets B, and $x, y \in Y$, then x and y can be separated by sets in the σ-algebra if and only if they can be separated by elements of the generating set.

($\Leftarrow$): Easy because elements of the generating set are elements of the σ-algebra.
($\Rightarrow$): Assume that the generating set fails to separate x and y. Consider the quotient space $Y' = Y/(x \sim y)$. Let $\mathcal{C}$ be some σ-algebra containing B. Then $\pi^{-1}\pi(\mathcal{C})$ contains B but fails to separate x and y, so the smallest σ-algebra containing B fails to separate x and y. So $\mathcal{B}$ fails to separate x and y.

Finally, let $[a] \in X/\sim$ be a point. I claim that $[a] \in \pi(\mathcal{A})$. For each point $[a] \neq [b] \in X/\sim$, $[a]$ and $[b]$ can be separated by elements of $\pi(\mathcal{A})$: otherwise $[a] = [b]$. So they can be separated by some element of the generating set. Define $\Lambda = \cap_{\{i\,|\,a\in A_i\}}\pi(A_i)$. Because there were only countably many A_i, this is the countable intersection of elements of $\pi(\mathcal{A})$; because we can separate non-equal points by elements of the generating set, $\Lambda = \{[a]\}$. So $\{[a]\} \in \pi(\mathcal{A}) \implies \{x\,|\,x \in [a]\} \in \mathcal{A}$ and is indecomposable, completing the proof.