Yoneda Embedding Small **Ban**-Enriched Categories in **Ban**

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Definition. A category $\mathcal{C}$ is said to be **Ban-enriched** if $\mathcal{C}(a, b)$ is a Banach space for each $a, b \in \text{Ob}(\mathcal{C})$, and composition from $\mathcal{C}(a, b) \times \mathcal{C}(b, c) \to \mathcal{C}(a, c)$ is bilinear and satisfies:

$$\|g \circ f\|_{\mathcal{C}(a,c)} \leq \|g\|_{\mathcal{C}(b,c)} \|f\|_{\mathcal{C}(a,b)}.$$

Question. Can every **Ban**-enriched category $\mathcal{C}$ be isometrically embedded into **Ban**, the category of Banach spaces?
Answer. Yes, assuming that the category $\mathcal{C}$ was small.

Throughout, let μ_c be the counting measure on $\text{Ob}(\mathcal{C})$. Define the functor $F : \mathcal{C} \to$**Ban** by the following rule: for objects,

$$F(a) = L^\infty(\text{Ob}(\mathcal{C}), \mathcal{C}(x, a), \mu_c(x))$$

(that is, the direct integral over $\text{Ob}(\mathcal{C})$ of the Hom spaces, which are Banach spaces, with respect to the supremum norm). Define it on morphisms by: if $f \in \mathcal{C}(a, b)$,

$$F(f) : F(a) \to F(b)$$
$$(\phi_x : x \to a)_{x \in \text{Ob}(\mathcal{C})} \mapsto (f \circ \phi_x : x \to b)_{x \in \text{Ob}(\mathcal{C})}$$

Let $(\phi_x) \in F(a)$, so there is an $M \in \mathbb{R}$ such that $\|\phi_x\| \leq M$ for each x. But $\|f \circ \phi_x\| \leq \|f\|\|\phi_x\| \leq \|f\|M$ for each x, so $(f \circ \phi_x)$ is an element of the L^∞ bundle. So this morphism is well-defined.
This is a bounded homomorphism of Banach spaces: it is linear by the linearity of composition, so it remains to prove boundedness. Without loss of generality, let $(\phi_x) \in F(a)$ have supremum norm 1.

$$\begin{aligned}
\|[F(f)](\phi_x)_{x \in \text{Ob}(\mathcal{C})}\| &= \|(f \circ \phi_x)\|_\infty \\
&= \sup_{x \in \text{Ob}(\mathcal{C})} \|f \circ \phi_x\| \\
&\leq \sup_{x \in \text{Ob}(\mathcal{C})} (\|f\|\|\phi_x\|) \\
&\leq \|f\| \sup_{x \in \text{Ob}(\mathcal{C})} \|\phi_x\| \\
&\leq \|f\|
\end{aligned}$$

In particular, applying $F(f)$ to the family

$$\phi_x = \begin{cases} 0 & x \neq a \\ \text{id}_a & x = a \end{cases}$$

we see that

$$[F(f)](\phi_x)_{x \in \mathrm{Ob}(\mathcal{C})} = \begin{cases} 0 & x \neq a \\ f & x = a \end{cases}$$

which transparently has ∞-norm $\|f\|$. So $\|F(f)\| = \|f\|$, so F is an isometric embedding of $\mathcal{C}(a,b)$ into $\mathbf{Ban}(F(a), F(b))$ for each $a, b \in \mathrm{Ob}(\mathcal{C})$.

It is trivial to check that $F(f) \circ F(g) = F(f \circ g)$, so F is an isometric embedding of $\mathcal{C}$ into **Ban**.

Note: It is possible to use the contravariant Yoneda embedding, defining the functor on objects by

$$G(a) = L^\infty(\mathrm{Ob}(\mathcal{C}), \mathcal{C}(a,x), \mu_c(x))$$

and on morphisms by

$$[G(f)](\phi_x)_{x \in \mathrm{Ob}(\mathcal{C})} = (\phi_x \circ f)_{x \in \mathrm{Ob}(\mathcal{C})}.$$

The arguments are quite similar to the above. If we want to fix the fact that the embedding was contravariant, it's easy to see that the Banach dual is a contravariant isometric embedding of **Ban** into itself, and composing the functors gives the requisite embedding.

Question: Are these two functors Banach-isomorphic?