Introductory Category Theory Notes

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0 Introduction

The following notes are currently being written for a graduate student learning seminar on category theory. The prerequisites for these notes are technically very minimal, but it helps to be familiar with some topology or algebra, since many of the examples are drawn from these fields. In addition some of the categorical constructions are motivated set-theoretically, the rudiments of set theory are assumed, but only at a very elementary level.

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1 What is a Category?

1.1 Basic Definitions and First Examples

A category is a collection of objects, together with a collection of composable morphisms between these objects. Note here that we say 'collection' to sidestep some set theoretic issues that we will take up later on.

Definition 1.1. A **category** $\mathcal{C}$ consists of a class of objects, denoted $\mathcal{O}(\mathcal{C})$, and, for every two objects $x, y \in \mathcal{O}(\mathcal{C})$, a class of morphisms, $\hom(x, y)$, satisfying the following properties:

(i) For every three objects $x, y, z \in \mathcal{O}(\mathcal{C})$, there is a composition law $\circ_{x,y,z} : \hom(y, z) \times \hom(x, y) \to \hom(x, z)$.

(ii) Composition is associative: for all $w, x, y, z \in \mathcal{O}(\mathcal{C})$, $f \in \hom(y, z), g \in \hom(x, y), h \in \hom(w, x)$ we have:

$$f \circ (g \circ h) = (f \circ g) \circ h$$

(iii) For each $x \in \mathcal{O}(\mathcal{C})$, there is a distinguished element $\mathrm{id}_x \in \hom(x, x)$ such that, for every $y \in \mathcal{O}(\mathcal{C})$,

- $f \circ id_x = f$ for all $f \in \hom(x, y)$
- $\mathrm{id}_x \circ f = f$ for all $f \in \hom(y, x)$.

There are three observations to make here, two notational and one foundational. The first notational point has to do with omitting information—we nearly always omit the subscripts from the composition symbol, or indeed we omit the symbol altogether, since a pair of morphisms is either composable, and then there is only one way of filling in the subscripts, or they are not, and then there isn't. Similarly, for compactness we will often avoid writing $x \in \mathcal{O}(\mathcal{C})$ for objects and just write $x \in \mathcal{C}$.

The second notational point is that we have to pay attention to morphisms—as we shall see there are numerous circumstances where we will care about distinguishing between categories with the same class of objects but different classes of morphisms, and so it will be confusing to simply write "hom." Some authors will write the category as a subscript, so that we have $\hom_{\mathcal{C}}(x, y)$ to denote "morphisms from x to y in the category $\mathcal{C}$" but we will opt for the more compact notation $\mathcal{C}(x, y)$.

The foundational issue is a bit stickier and this has to do with the fact that we talk of "classes" of objects and morphisms rather than "sets." The motivation for this is that we would like to speak of Set the category of all sets with morphisms functions between sets, and we could not do this if we required a "set" of objects, since this would then require a set of all sets, which as we all know doesn't exist. One common intution is that there are "too many" sets to fit them all in one set, and this motivates the following terminology: a category is called **small** if the class of objects is a set, and it is called **locally small** if the class of morphisms between any two objects is a set.

Since we have already mentioned an example, Set, let's take a moment to examine it. In this category objects are sets, morphisms are functions between sets, and the associativity of the composition law is the associativity of composition of functions. Formally:

Example 1.2 (The category of sets)**.** Define Set as follows: $\mathcal{O}(\mathsf{Set})$ is the class of all sets, and, for any two sets $A, B \in \mathcal{O}(\mathsf{Set})$, define $\hom(A, B) = \{f : A \to B\}$ as the set of functions from A to B, with the composition law given by the usual composition of functions. Since composition of functions is associative, and there is always an identity function, Set is a category.

Many categories can be found living inside the category of sets – objects are sets equipped with certain structures, and morphisms are still functions. Here are a few:

The category	Objects	Morphisms
Top	Topological spaces	Continuous functions
Man	Topological manifolds	Continuous functions
$\mathsf{SmoothMan}$	Smooth manifolds	Smooth functions
FI	Sets	Injective functions
FS	Sets	Surjective functions
FB	Sets	Bijective functions
k-Vect	k-vector spaces	k-linear functions
R-Mod	Left R-modules	R-linear functions
Ban	Banach spaces	Bounded linear functions
$\mathsf{BanShort}$	Banach spaces	Linear functions of norm at most 1

Each of these is what we call a subcategory of the category of sets.

Definition 1.3 (Subcategory)**.** A **subcategory** $\mathcal{C}'$ of a category $\mathcal{C}$ is a subcollection of objects of $\mathcal{O}(\mathcal{C})$, which we denote $\mathcal{O}(\mathcal{C}')$, and, for every $x, y \in \mathcal{C}'$, a subclass $\mathcal{C}'(x, y) \subseteq \mathcal{C}(x, y)$ which contains the identity morphsim for each object and is closed under composition. A subcategory is naturally a category under the inherited composition law and choice of identity.

Definition 1.4 (Cantorian)**.** A category which is a subcategory of Set is sometimes called **cantorian**. A category in which the objects are sets but the morphisms are *not* functions is sometimes called **quasi-cantorian**.

It is quite hard to construct an example of a category which is not cantorian, and harder still to construct one which is not at least quasi-cantorian.

Remark 1.5. It is very common in the literature for people to define a category simply by stating the objects and not the morphisms, for example we might have said "Set is the category whose objects are sets" but as evidenced by the table above this does not uniquely specify a category. Ironically, instead of forgetting the morphisms, we can usually forget the objects in a category instead—it would make more sense to say that we want to look at the category where morphisms are functions from sets to sets, and this does uniquely determine a category. More formally, we could define a category to be simply a collection of morphisms following certain axioms about composition, and the identity arrows would allow us to recover the notion of an object. This project is left as an exercise to the reader, who could also consult [3] for more details. This may seem like mere sophistry but this notion of interchageability of an object with the identity morphism on that object is an important step in developing categorical intution.

1.2 Thinking in Diagrams

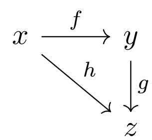
There are many different sorts of notation used in category, but the most ubiquitous is the commutative diagram. A diagram in a category is a graph like the one below:

$$\begin{CD} x @>f>> y \\ @VgVV @VVhV \\ z @>j>> w \end{CD}$$

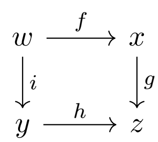
The vertices in the graph are labelled by objects in the category, and the edges are labelled by morphisms (or 'arrows') in the category. Generally the diagram doesn't contain all of the implied morphisms, for example there is of course a morphism from x to w by $h \circ f$ and one from $j \circ g$. These are left out so as not to clutter the diagram. In a diagram, any path of arrows from one object to another gives rise to a morphism through composition. When we draw a diagram it is almost always to claim that all such compositions are the same. More concretely:

Notation 1.6. A diagram is commutative if for every pair of objects, all paths from one to the other give rise to the same morphism.

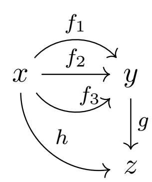
For example,



means that $f : x \to y$, $g : y \to z$, $h : x \to z$ are morphisms in the assumed category $\mathcal{C}$, and $g \circ f = h$. Similarly,



means that $h \circ i = g \circ f$, and



means that $g \circ f_1 = g \circ f_2 = g \circ f_3 = h$. That is, the statement that a diagram commutes expresses equality of certain morphisms.

Convention 1.7. Unless explicitly stated otherwise, in these notes (and in almost every text with which the authors are familiar) a diagram is commutative unless explicitly stated otherwise.

1.3 The Language of Morphisms

"It is better to have a good category with bad objects than a bad category with good objects." – Grothendieck, apocryphal.

When one is talking about sets, one can talk about elements of those sets. When one is talking about the category Set one can no longer talk about elements of sets as each set is simply an object in Set and objects in categories have nothing "inside" of them. The only tool at our disposal is the morphism. This may at first seem like a drawback, but the shift in perspective it forces is a valuable one. In this section we will start to build up some of the tools for working with morphisms and show how they replace what we would otherwise do with elements. Those who know more category theory may point out that we do in fact have a notion of "element" of a set, but this definition is also entirely defined in terms of morphisms (essentially, an element is a morphism from the set with one element, which can be singled out using its relationship to morphisms... we will get there.)

Of course, one may point out in the world of sets that if elements cannot be distinguished, there is no separation allowed between bijective sets. This is of course correct, and lends a certain primacy to the notion of *isomorphism*.

Definition 1.8. In a category $\mathcal{C}$, two objects $x, y \in \mathcal{C}$ are **isomorphic** (to each other) if there are $f \in \mathcal{C}(x, y), g \in \mathcal{C}(y, x)$ such that $gf = \mathrm{id}_x$ and $fg = \mathrm{id}_y$. In this case, we say that f and g are **isomorphisms**, and write $x \cong y$.

Notation 1.9. If f is an isomorphism, the morphism g such that $gf = fg = \mathrm{id}$ is uniquely determined. We write $g = f^{-1}$, and say that g is the inverse of f.

Different categories will have radically different notions of isomorphism. In the category of topological spaces, isomorphism is called homeomorphism; in the category of metric spaces with non-expansive maps, it is called isometry. Algebraists generally use the word "isomorphism" to refer to isomorphism in all of their categories. In any case, the language of category theory is generally unable to distinguish between isomorphic objects. It is often the case that we cannot determine objects on the nose, but merely "up to isomorphism."

Now that we have identified the bijections in Set it may seem like a natural next step to ask about injections (or surjections, which we will get to next.)We start with a pair of sets A, B and a morphism $f : A \to B$. We want to tell if f is injective, but we can't pick elements in A and "look at where f sends them." How then can we talk about "distinguishing elements?" Consider an arbitrary set C and a set of parallel arrows $g, h : C \to A$ so that the diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & A \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

commutes. Then what can we say about g, h? If C is a set with a single element, then g and h pick out elements of A, and if f is injective then the statement that the diagram commutes (i.e. that f sends both of these elements to the same place) means that these elements must be the same. Further, this will still hold no matter the cardinality of C–if g and h ever send the same element to a different pair of elements in A, then the injectivity of f means that these elements will still be distinguished in B. Then f is injective if for *every* pair of parallel arrows g, h the above diagram commutes only if g and h are the same. This motivates the following definition.

Definition 1.10. A morphism $f \in \mathcal{C}(x, y)$ is called a **monomorphism** if it satisfies the following property:
for every $w \in \mathcal{C}$, $g_1, g_2 \in \mathcal{C}(w, x)$, $fg_1 = fg_2 \implies g_1 = g_2$. This property is called left cancellation.

Notation 1.11. When f is a monomorphism, we often use $\hookrightarrow$ or $\rightarrowtail$ to denote it. So the injection $i : \mathbb{Q} \to \mathbb{R}$ can be written $i : \mathbb{Q} \hookrightarrow \mathbb{R}$.

We strongly caution the reader that while injectivity can be viewed as a motivation for the notion of a monomorphism, these are *not* the same. In a subcategory of Set isomorphisms are always bijective, but monomorphisms are not always injective.

Example 1.12. We begin with an example for the algebraically inclined. Consider the category of divisible abelian groups, with morphisms being additive functions, and the projection $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. This is clearly not injective, and yet it is a monomorphism. Indeed, consider another divisible abelian group A, and a pair of morphisms f, g so that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & \mathbb{Q} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \pi} \\ \mathbb{Q} & \xrightarrow{\pi} & \mathbb{Q}/\mathbb{Z} \end{array}$$

commutes. Then we want to show $f = g$. Suppose for contradiction that $f \neq g$. Since we can add and subtract maps of abelian groups replace in the above diagram f by $f - g$ and g by $g - g$, i.e. the 0 map. This new diagram is still commutative (why?) and so we now have a diagram

$$\begin{array}{ccc} A & \xrightarrow{h=f-g} & \mathbb{Q} \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \pi} \\ \mathbb{Q} & \xrightarrow{\pi} & \mathbb{Q}/\mathbb{Z} \end{array}$$

and since $f \neq g$ we know that $h \neq 0$. By commutativity of the diagram $\pi \circ h$ factors through the 0 map, and so to reach a contradiction we will locate an element of A which is not in the kernel of $\pi \circ h$. Since $h \neq 0$ we know there is some element a not in the kernel of h. Then if $h(a)$ is not in the kernel of π (i.e. an integer) we have a contradiction, so we suppose that $h(a)$ is an integer k. Then we pick another integer j which does not divide k and consider $\frac{a}{j}$ in A which must be sent to $\frac{k}{j}$ by h, and since this is not an integer it is not in the kernel of π. Then $\frac{a}{j}$ is not in the kernel of $\pi \circ h$ and we have a contradiction. Then $f = g$.

Example 1.13. For the more topologically inclined, consider the category of path-connected, locally path-connected pointed topological spaces (recall that a *pointed* space is a space together with a choice of basepoint). Then let X be such a space and $\pi : \tilde{X} \to X$ a covering map. This covering map may or may not be injective (in general it will not be) but it is always a monomorphism. The key piece of insight here is that we have restricted to the case where we can lift maps to X back up to the covering space $\tilde{X}$. Any map f which can be factored through $\tilde{X}$ as some map $\pi \circ \tilde{f}$ can be lifted, and since we have fixed the basepoint it can be lifted uniquely back to f. Then pick arrows g, h from some space Y to $\tilde{X}$. Since these maps are both lifts of the maps $\pi \circ g = \pi \circ h$, uniqueness of lifts gives $g = h$.

We hope that with this pair of examples it is clear to the reader that the concept of monomorphism and the concept of injection are not the same, although one has motivated the definition of the other. We now turn to surjectivity.

Again we want to think about sets A, B and a morphism $f : A \to B$ and ask how we can tell if it is surjective purely in terms of arrows. For injectivity we wanted to somehow carry over the idea that f can "see the difference" between any elements in A, but here we want to carry over the idea that f "sees" all the elements of B. We then consider a pair of parallel arrows $g, h : B \to C$ so that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow{h} & C \end{array}$$

commutes. A very similar analysis to the one above for injective functions leads to the following definition.

Definition 1.14. A morphism $f \in \mathcal{C}(x, y)$ is called a **epimorphism** if it satisfies the following property:

for every $z \in \mathcal{C}$, $h_1, h_2 \in \mathcal{C}(y, z)$, $h_1 f = h_2 f \implies h_1 = h_2$. This property is called right cancellation.

Notation 1.15. When f is an epimorphism, we often use $\twoheadrightarrow$ to represent it. So the surjection $\mathbb{Z} \to \mathbb{Z}/(2)$ can be be written $\pi : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/(2)$.

Again, though we have motivated epimorphisms with the idea of surjectivity, and in Set these are the same, they are not the same in all Cantorian categories.

Example 1.16. Let $\mathcal{C}$ be **TopHaus**, the category of Hausdorff topological spaces. Let $i : \mathbb{Q} \to \mathbb{R}$ be the standard inclusion. Then if X is another Hausdorff topological space, $h_1, h_2 : \mathbb{R} \to X$ continuous functions with $h_1 i = h_2 i$. Then $h_1 = h_2$, based on the following sketch argument: for any $x \in \mathbb{R}$, $x = \lim\limits_{n\to\infty} q_n$, $q_n \in \mathbb{Q}$. Then:

$$
\begin{aligned}
h_1(x) &= h_1(\lim_{n\to\infty} x_n) \\
&= \lim_{n\to\infty} h_1(x_n) \\
&= \lim_{n\to\infty} h_2(x_n) \\
&= h_2(\lim_{n\to\infty} x_n) \\
&= h_2(x)
\end{aligned}
$$

using throughout that in Hausdorff spaces, convergent sequences have at most one limit, and that continuous functions preserve limits. However, the injection from $\mathbb{Q}$ to $\mathbb{R}$ is not a surjection, even though it is an epimorphism.

Example 1.17 (Best if you know sheaves)**.** In the category of sheaves on the punctured disk $\mathring{\mathbb{D}}$, the differentiation map $\partial : \mathcal{O} \to \mathcal{O}$ is an epimorphism, but $\frac{1}{z} \in \mathcal{O}$ is not the derivative of any analytic function. Here $\mathcal{O}$ represents the sheaf of analytic functions.

Example 1.18. Consider the category of torsion free abelian groups, and the inclusion morphism $i : Z \to \mathbb{Q}$. This is an epimorphism, although clearly not surjective. Consider a pair of arrows $f, g : \mathbb{Q} \to A$ to some torsion free abelian group A with the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\;i\;} & \mathbb{Q} \\
{\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
\mathbb{Q} & \xrightarrow{\;f\;} & A
\end{array}
$$

Again we consider the modified diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{i} & \mathbb{Q} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 0} \\ \mathbb{Q} & \xrightarrow{f-g} & A \end{array}$$

and set $h = f - g$. Again commutativity tells us that since $h \circ i$ factors through the 0 map it must be the 0 map. Then the kernel h must contain the image of i. Without loss of generality, replace A with the image of h. Since A is isomorphic to $\mathbb{Q}/ker(h)$ and it is torsion free, we conclude that if $\mathbb{Z} \subset ker(h)$ that $ker(h)$ must be all of $\mathbb{Q}$, which means h is the 0 map, or $f = g$.

Remark 1.19. In Set a bijection is a map which is both surjective and injective. In general categories the situation is more complicated. If f is an isomorphism, then f is both a monomorphism and an epimorphism. To show it is a monomorphism, fix an inverse g for f, and assume that $h_1 f = h_2 f$. Then $h_1 fg = h_2 fg = h_1 \mathrm{id} = h_2 \mathrm{id}$, so $h_1 = h_2$. A similar argument shows that it is an epimorphism. The reverse however needn't be the case. In fact, all of the examples above, except for sheaves on the punctured disk, demonstrate this.

It's clear that being an isomorphism is stronger than being a monomorphism or an epimorphism. Lying somewhere in between these two concepts are sections and retractions, also right and left inverses.

Definition 1.20. A **section** is a morphism $f \in \mathcal{C}(x, y)$ such that there is some morphism $g \in \mathcal{C}(y, x)$ with $gf = \mathrm{id}_x$. Sometimes we say that f is a **section of** g, or a right inverse to g. Given g, if such an f exists we say that g **admits a section**. All sections are monomorphisms.

Definition 1.21. A **retraction** is a morphism $g \in \mathcal{C}(x, y)$ such that there is some morphism $f \in \mathcal{C}(y, x)$ with $gf = \mathrm{id}_x$. Sometimes we say that f is a **retraction of**, or left inverse to g. Given g, if such an f exists we say that g **admits a retraction**. All retractions are epimorphisms.

If a morphism is both a section and a retraction, then it is an isomorphism. In Set all monomorphisms are sections and all epimorphisms are retractions (assuming the axiom of choice), but again this needn't be the case in an arbitrary Cantorian category, although it is true in any category with an epi-mono factorization (we may prove this later, but for now do not worry about it.)

1.4 Functors

To close out this chapter, we will learn about one more kind of morphism—a morphism of categories, i.e. a functor.

Definition 1.22. A **functor** from a category $\mathcal{C}$ to a category $\mathcal{D}$, normally denoted $F : \mathcal{C} \to \mathcal{D}$, consists of the following data:

(i) For each object $x \in \mathcal{C}$, an object $F(x) \in \mathcal{D}$.

(ii) For each morphism $f \in \mathcal{C}(x, y)$, a morphism $F(f) \in \mathcal{D}(F(x), F(y))$.

(iii) Respecting composition: for composable f, g, $F(g \circ f) = F(g) \circ F(f)$.

(iv) Respecting identity: $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

A functor takes objects to objects, morphisms to morphisms, and commuting diagrams to commuting diagrams. Technically, what we have defined here is a **covariant functor**. It contrasts with the following:

Definition 1.23. A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ assigns to each object $x \in \mathcal{C}$ an object $F(x) \in \mathcal{D}$, and to each morphism $f \in \mathcal{C}(x, y)$, a morphism $F(f) \in \mathcal{D}(F(y), F(x))$, obeying $F(g \circ f) = F(f) \circ F(g)$, and $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

A covariant functor flips the direction of the arrows, which is something that comes up a lot in category theory.

Functors are all over the place! Here are a few examples:

Example 1.24 (Continuous functions)**.** Let **Top** denote the category of topological spaces, and $\mathbb{C}$-**Vect** the category of $\mathbb{C}$-vector spaces. Define $F : \mathbf{Top} \to \mathbb{C}-\mathbf{Vect}$ by $F(X) = \{f : X \to \mathbb{C} \text{ continuous}\}$. If $g : X \to Y$ is a continuous function, it induces a linear map $F(Y) \to F(X)$ by composition: if $f : Y \to \mathbb{C}$ is continuous, then $f \circ g : X \to \mathbb{C}$ is continuous. Check that this is linear and so on. In fact, this map respects the multiplication of functions, so it's a functor to the category of $\mathbb{C}$-algebras.

Example 1.25 (π_1)**.** Let $\mathcal{C}$ denote the category of pointed topological spaces: objects are topological spaces with distinguished basepoint, (X, x_0), and morphisms are continuous functions that take basepoint to basepoint. A morphism in this category $f : (X, x_0) \to (Y, y_0)$ induces a map $\pi_1(f) = f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Common Functors

Functor	Source category	Target category	Action	Variance
$\mathcal{C}(-,x)$	Any locally small category $\mathcal{C}$	**Set**	Takes any object y to the set of morphisms $\hom_{\mathcal{C}}(y,x)$; composes by precomposition	Contra-
π_k	$\mathbf{Top}_*$	**Grp**	Takes a pointed space X to homotopy classes of maps of k-spheres into it	Co-
Spec	**ComRing**	**Top**	Takes a commutative ring to its prime ideals with the Zariski topology	Contra-

A functor is the right notion of morphism of categories. In this context, we can define an isomorphism of categories:

Definition 1.26. Two categories $\mathcal{C}$ and $\mathcal{D}$ are **isomorphic** if there are functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ such that GF is the identity functor on $\mathcal{C}$, and FG is the identity functor on $\mathcal{D}$.

Just like isomorphism inside a category, isomorphism of categories preserves many important qualities. However, isomorphism of categories turns out to be too strong of a notion, and we will eventually accept the weaker condition of equivalence.

1.5 Exercises

1. Write down your favorite category. What are the objects? What are the morphisms? Find a subcategory by picking a particularly nice family of morphisms.

2. Given a partially ordered set, create a category where the objects are the elements of the set, and the morphisms are determined by inclusions. Are these morphisms monomorphisms? Epimorphisms?

3. Identify the monomorphisms in the following categories:

    (i) The category of groups, with group homomorphisms as morphisms.

(ii) The category of rings, with ring homomorphisms as morphisms.

(iii) The category of partially ordered sets, with order-preserving functions as morphisms.

(iv) The category of topological spaces, with continuous maps as morphisms.

4. Show that an isomorphism has a unique inverse.

5. Show that a morphism which is both a section and a retraction is an isomorphism.

6. Prove the statement or give a counterexample (assuming all functors covariant):

    (i) Functors always take monomorphisms to monomorphisms.

    (ii) Functors always take epimorphisms to epimorphisms.

    (iii) Functors always take sections to sections.

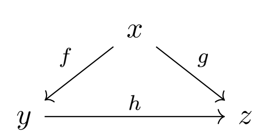
    (iv) Functors always take retractions to retractions.

    (v) Functors always take isomorphisms to isomorphisms.

7. Find an example of a retract of categories: categories $\mathcal{C}$ and $\mathcal{D}$ and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ with GF being the identity functor on $\mathcal{C}$. Bonus if FG is not the identity on $\mathcal{D}$.

8. Fix a category $\mathcal{C}$ and an object $x \in \mathcal{C}$. Define the following category, sometimes referred to as the x-pointed category of $\mathcal{C}$:

    - Objects are pairs (y, f), with $y \in \mathcal{C}$, $f \in \mathcal{C}(x, y)$.
    - A morphism from (y, f) to (z, g) is an element $h \in \mathcal{C}(y, z)$ such that the following diagram commutes in $\mathcal{C}$:



    with the composition inherited from $\mathcal{C}$.

Prove that this category, $x \downarrow \mathcal{C}$, actually is a category. Show that there is a canonical functor $F : x \downarrow \mathcal{C} \to \mathcal{C}$. What would it mean if F was an isomorphism of categories? What would that imply about the functor $\mathcal{C}(x, -)$? Find some examples of such categories.

9. Find an example of two categories $\mathcal{C}$ and $\mathcal{D}$, with a functor $F : \mathcal{C} \to \mathcal{D}$ such that F takes some non-isomorphic objects in $\mathcal{C}$ to isomorphic objects in $\mathcal{D}$.

10. Given a category $\mathcal{C}$ and an object $x \in \mathcal{C}$, show that the set of invertible morphisms $x \to x$, denoted $\mathrm{Aut}(x)$, is a group.

11. Establish the following equivalences:

    (i) A group is a one-object locally small category where every morphism is invertible.

    (ii) A monoid is a one-object locally small category.

12. Given a locally small category $\mathcal{C}$ and an object $x \in \mathcal{C}$, build a canonical poset from $\mathcal{C}$ by taking the elements of the poset to be the idempotents in $\mathcal{C}(x, x)$, and declaring $f \leq g$ if $gf = f$. Given any poset, can you build a category where the idempotents correspond exactly to the poset in this manner?

13. Show that the composition of monics is a monic.

14. Suppose you have arrows f, g whose composition gf is a monomorphism. Is f always a monomorphism? Is g?

15. Show that in a category $\mathcal{C}$, the relation $x \simeq y \iff$ there is an isomorphism between x and y defines an equivalence relation on the objects of $\mathcal{C}$.

16. Define a category $\mathcal{C}$ to be **skeletal** if for every pair of objects $x, y \in \mathcal{C}$, $x \cong y \iff x = y$. Given a small category $\mathcal{D}$, construct its skeletalization: find a skeletal category $\mathcal{C}$ and a functor $F : \mathcal{D} \to \mathcal{C}$ which is an equivalence of categories: there is a $G : \mathcal{C} \to \mathcal{D}$ with $GF \cong \mathrm{id}_{\mathcal{D}}, FG \cong \mathrm{id}_{\mathcal{C}}$.

17. Define the category FinOrd to be the category with objects all sets of the form $\{0, \ldots, n-1\}$ for $n \in \mathbb{N}$, and with morphisms all functions between these sets. Show that FinOrd is skeletal, and, moreover, that it is a skeletalization of the category of finite sets.

2 Natural Transformations, Categories of Functors

2.1 Natural Transformations

Definition 2.1. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a **natural transformation** from F to G is a collection of maps $(\eta_x)_{x \in \mathcal{C}}$, where each map η_x is in $\mathcal{D}(Fx, Gx)$. The naturality condition is expressed by the following commuting square: for any $f \in \mathcal{C}(x, y)$,

$$\begin{array}{ccc} Fx & \xrightarrow{Ff} & Fy \\ \downarrow{\scriptstyle \eta_x} & & \downarrow{\scriptstyle \eta_y} \\ Gx & \xrightarrow{Gf} & Gy \end{array}$$

Example 2.2. Let $\mathcal{C}$ be the category of sets. For each set S, there is a functor $- \times S$, defined by the following: on objects, a set A maps to the set $A \times S$, its Cartesian product; on morphisms, a function $f : A \to B$ maps to the function

$$\begin{aligned} f \times \mathrm{id}_S : A \times S &\to B \times S \\ (a, s) &\mapsto (f(a), s) \end{aligned}$$

(you should check that this functor preserves commuting triangles). Then for any function $g : S \to T$, $\mathrm{id}_{-} \times g : - \times S \to - \times T$ is a natural transformation between the two functors. To check it is a natural transformation, we first pick two objects A, B, and a morphism $f : A \to B$. We want to check that the following diagram commutes:

$$\begin{array}{ccc} A \times S & \xrightarrow{f \times \mathrm{id}_S} & B \times S \\ \downarrow{\scriptstyle \mathrm{id}_A \times g} & & \downarrow{\scriptstyle \mathrm{id}_B \times g} \\ A \times T & \xrightarrow[f \times \mathrm{id}_T]{} & B \times T \end{array}$$

Pick a pair (a, s) in the top left corner. Going right, then down, maps $(a, s) \mapsto (f(a), s) \mapsto (f(a), g(s))$; going down, then right, maps $(a, s) \mapsto (a, g(s)) \mapsto (f(a), g(s))$. So the diagram commutes for arbitrary A, B, f; we have built a natural transformation.

Remark 2.3. It's nice to note that in some sense, the natural transformation commutes the two functors. If $\eta : F \to G$ is a natural transformation, for any $f \in \mathcal{C}(x, y)$, $G(f) \circ \eta_x = \eta_y \circ F(f)$. Removing labels, we can say that $G\eta = \eta F$.

Example 2.4 (Double dualization)**.** One of the most recognized natural transformations is the evaluation homomorphism $V \to V^{**}$, where V^* is the dual of a vector space. To be precise, fix a field k, and let $\mathcal{C}$ denote the category of k-vector spaces. The functor commonly denoted $*$, or dualization, takes a vector space V to $\mathrm{Hom}_k(V, k)$, and if $T : V \to W$ is a k-linear map, by means of the following diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ & \underset{\phi\circ T}{\searrow} & \downarrow{\scriptstyle \phi} \\ & & k \end{array}$$

T induces a map $T^* : W^* \to V^*$. Checking some coherence (linearity of $T*$, respecting composition and identity) we see that this is a functor. Now we can define a natural transformation from the identity functor to the functor $**$. For each V, the component should be

$$\begin{aligned} \iota_V : V &\to V^{**} \\ v &\mapsto (\phi \mapsto \phi(v)) \end{aligned}$$

for each $\phi \in V^*$. This is evaluation of the linear functionals at v.

Naturality should be expressed by the following square: for any $V, W \in \mathcal{C}$, $T \in \mathcal{C}(V, W)$, we should have

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow{\scriptstyle \iota_V} & & \downarrow{\scriptstyle \iota_W} \\ V^{**} & \xrightarrow{T^{**}} & W^{**} \end{array}$$

Again, we start at the top left, and take both paths. Going right, then down, takes an element $v \in V$ first to $T(v) \in W$, and then to the map which takes any $\phi : W \to k$ to $\phi(T(v))$. Going down, then right, takes v first to the functional ev_v that takes any map $\psi : V \to k$ to $\psi(v)$, and then to $T^{**}(\mathrm{ev}_v)$. By the definition of the functor $*$, $T^{**}(\mathrm{ev}_v) = \mathrm{ev}_v \circ T^*$. For any $\phi \in V^*$, $\mathrm{ev}_v \circ T^*(\phi) = \mathrm{ev}_v(\phi \circ T) = \phi(T(v))$, exactly as desired.

Example 2.5. Let G be a group, and let $G-\mathsf{Set}$ be the category where objects are are sets with left actions by G, $\rho : G \to \mathsf{Set}(X, X)$, and morphisms are functions $f : (X, \rho) \to (Y, \sigma)$ such that for each $g \in G$, $f(\rho(g)(x)) = \sigma(g)(f(x))$. Let U denote the forgetful functor $G - \mathsf{Set} \to \mathsf{Set}$. Then each $g \in G$ gives a natural transformation $\eta^g : U \to U$ by the following rule: $\eta^g_{(X,\rho)}(x) = \rho(g)(x)$. This is a natural transformation by fiat – we declared that morphisms in the category respected the group action.

Of course, we can now say that two functors are naturally isomorphic if there are natural transformations between them that compose to the identity both ways.

The structure of categories, with morphisms being functors, and morphisms between functors being natural transformations, is the basis of 2-categories.

2.2 Functor Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We can form a new category from the functors between them.

Definition 2.6. If $\mathcal{C}$ and $\mathcal{D}$ are categories, the **functor category** from $\mathcal{C}$ to $\mathcal{D}$, which we normally denote $[\mathcal{C}, \mathcal{D}]$, is the category with:

1. Objects are functors $F : \mathcal{C} \to \mathcal{D}$;

2. Morphisms between functors are natural transformations $\eta : F \to G$.

Example 2.7. Let G be a group, considered as the morphisms of the one-object category $\mathcal{C}$. Then the functor category $[\mathcal{C}, \mathcal{D}]$ is the category with objects being objects in $\mathcal{D}$ together with G-symmetries, and morphisms being morphisms in $\mathcal{D}$ which respect the group action. In particular, $G - \mathsf{Set} \cong [\mathcal{C}, \mathsf{Set}]$.

To ask for a category to have a bit of extra structure, we can phrase the request in terms of a functor.

Functor categories are very important for the Yoneda lemma, limits, and colimits.

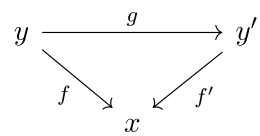
2.3 Slice and Coslice Categories

A slice or coslice category is a category equipped with the data of a particular morphism from a particular object.

Definition 2.8. Given a category $\mathcal{C}$ and an object $x \in \mathcal{C}$, the **slice category**, sometimes called the **slice category over** x, has

1. Objects are pairs (y, f), with $f \in \mathcal{C}(y, x)$;

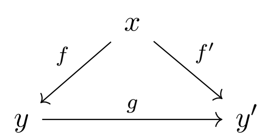
2. Morphisms from (y, f) to (y', f') are those $g \in \mathcal{C}(y, y')$ with the following diagram commuting:



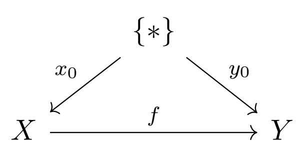
Definition 2.9. Given a category $\mathcal{C}$ and an object $x \in \mathcal{C}$, the **coslice category**, sometimes called the **coslice category under** x, has

1. Objects are pairs (y, f), with $f \in \mathcal{C}(x, y)$;

2. Morphisms between (y, f) and (y', f') are those morphisms $g \in \mathcal{C}(y, y')$ with the following diagram commuting:



Example 2.10. The category of pointed topological spaces is naturally a coslice category. Recall that the category of pointed topological spaces has objects being pairs (X, x_0), and a morphism from (X, x_0) to (Y, y_0) is a continuous function f with $f(x_0) = y_0$. This can be expressed in the following commutative diagram:



which is transparently the coslice category under a point.

Example 2.11. The category of k-schemes is the slice category of schemes over $\operatorname{Spec} k$.

2.4 Comma Categories

Everything in the above can be generalized in the form of a comma category.

Definition 2.12. Given three categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$, with two functors $F : \mathcal{C} \to \mathcal{E}$, $G : \mathcal{D} \to \mathcal{E}$, the **comma category** is defined as having

1. Objects are triples (c, d, α) with $c \in \mathcal{C}$, $d \in \mathcal{D}$, and $\alpha \in \mathcal{E}(Fc, Gd)$;

2. Morphisms from (c, d, α) to (c', d', α') are pairs (β, γ) with $\beta \in \mathcal{C}(c, c')$, $\gamma \in \mathcal{D}(d, d')$, such that the following square commutes:

$$\begin{array}{ccc} Fc & \xrightarrow{F\beta} & Fc' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ Gd & \xrightarrow{G\gamma} & Gd' \end{array}$$

3. Composition of morphisms is given by composition on their components.

The comma category is a very powerful, general notion.

Example 2.13. If $\mathcal{E} = \mathcal{C}$ and $F = \mathrm{id}_{\mathcal{C}}$, and $\mathcal{D}$ is the category with just one object and no nontrivial morphisms, with $G(*) = c$, then the comma category is the slice category over c.

Example 2.14. If, conversely, $G = \mathrm{id}_{\mathcal{C}}$ and F is the inclusion of an object $c \in \mathcal{C}$, then we have the coslice category under c.

2.5 Exercises

1. Prove Example 2.13.

2. Prove 2.14.

3. If $F_1, F_2, F_3 : \mathcal{C} \to \mathcal{D}$, and $\eta_1 : F_1 \to F_2, \eta_2 : F_2 \to F_3$, then $\eta_2 \circ h_1 : F_1 \to F_3$. Show that $\eta_2 \circ \eta_1$ is a natural transformation.

4. If $F_1, F_2 : \mathcal{C} \to \mathcal{D}$, $G_1, G_2 : \mathcal{D} \to \mathcal{E}$, $\eta_1 : F_1 \to F_2$, $\epsilon_1 : G_1 \to G_2$, define a horizontal composition $\epsilon_1\eta_1 : G_1F_1 \to G_2F_2$ by the following rule: for each $x \in \mathcal{C}$, compose the maps

$$\begin{array}{ccc} G_1F_1(x) & \xrightarrow{G_1((\eta_1)_x)} & G_1F_2(x) \\ {\scriptstyle (\epsilon_1)_{F_1(x)}}\downarrow & & \downarrow{\scriptstyle (\epsilon_1)_{F_2(x)}} \\ G_2F_1(x) & \xrightarrow[(\epsilon_1)_{F_1(x)}]{} & G_2F_2(x) \end{array}$$

to get from G_1F_1 to G_2F_2. Show that both compositions are the same, and that they assemble to a natural transformation $G_1F_1 \to G_2F_2$. This is the horizontal composition of natural transformations.

5. Show that if $F_1, F_2, F_3 : \mathcal{C} \to \mathcal{D}$, $G_1, G_2, G_3 : \mathcal{D} \to \mathcal{E}$, and $\eta_1 : F_1 \to F_2, \eta_2 : F_2 \to F_3$, $\epsilon_1 : G_1 \to G_2$, $\epsilon_2 : G_2 \to G_3$, then vertical and horizontal compositions commute:

$$(\epsilon_2 \circ \epsilon_1)(\eta_2 \circ h_1) = (\epsilon_2\eta_2) \circ (\epsilon_1\eta_1).$$

6. Fix a base field k. Consider the two functors $k - \mathsf{Mod}^{op} \times k - \mathsf{Mod} \to k - \mathsf{Mod}$ by $F(-_1, -_2) = -_1^* \otimes -_2$ and $G(-_1, -_2) = \mathrm{Hom}_k(-_1, -_2)$. Construct a natural transformation $F \to G$, and show that when restricted to the category of finite-dimensional vector spaces, this natural transformation is a natural isomorphism. Construct its inverse in that case.

3 The Yoneda Lemma

The Yoneda Lemma is a canonical embedding of a locally small category $\mathcal{C}$ into the category $[\mathcal{C}^{op}, \mathsf{Set}]$, generally known as the category of presheaves on $\mathcal{C}$, $\mathsf{PSh}(\mathcal{C})$.

Throughout this section, we use the notation Nat to denote the class of natural transformations from one functor (particularly presheaf) to another, instead of our usual $[\mathcal{C}^{op}, \mathsf{Set}]$.

Definition 3.1. A functor $F : \mathcal{C} \to \mathcal{D}$ is called **full** if for every pair of objects $x, y \in \mathcal{C}$, the map $F_{x,y} : \mathcal{C}(x, y) \to \mathcal{D}(x, y)$ induced by the functor is surjective.

Definition 3.2. A functor $F : \mathcal{C} \to \mathcal{D}$ is called **faithful** if for every pair of objects $x, y \in \mathcal{C}$, the map $F_{x,y} : \mathcal{C}(x, y) \to \mathcal{D}(x, y)$ induced by the functor is injective.

Remark 3.3. It is tempting to claim that a functor that is both full and faithful is an equivalence of categories, in analogy with algebraic objects. However, this is not correct. In order for F to be an equivalence of categories, it must also be **essentially surjective** – for each $z \in \mathcal{D}$, there must be some $x \in \mathcal{C}$ such that $F(x) \cong_{\mathcal{D}} z$.

Theorem 3.4 (Yoneda)**.** Let $\mathcal{C}$ be a locally small category. Define the functor $h : \mathcal{C} \to \mathsf{PSh}(\mathcal{C})$ by:

1. For each object $x \in \mathcal{C}$, $x \mapsto h^x = \mathcal{C}(-, x)$.
2. For each $f \in \mathcal{C}(x, y)$, f maps to the natural transformation $\mathcal{C}(-, x) \to \mathcal{C}(-, y)$ by $f \circ -$.

Then h is a full and faithful functor. Moreover, for any $F \in [\mathcal{C}^{op}, \mathsf{Set}]$, $\mathsf{Nat}(h^x, F) \cong F(x)$.

Definition 3.5. A functor of the form $h^x = \mathcal{C}(-, x)$, or of the form ${}^x h = \mathcal{C}(x, -)$, is called **representable**.

Proof. Checking that h is actually a functor is easy. Certainly, it sends the identity morphism to the identity natural transformation, since $\mathrm{id}_x \circ - = -$. If $g \in \mathcal{C}(x, y), f \in \mathcal{C}(y, z)$, then

$$\begin{aligned}(h^f \circ h^g)(-) &= h^f(g \circ -) \\ &= f \circ (g \circ -) \\ &= (f \circ g) \circ - \\ &= h^{f \circ g}(-).\end{aligned}$$

The fact that $f \circ -$ defines a natural transformation is expressed in the following commutative square: let $f \in \mathcal{C}(x, y)$, $g \in \mathcal{C}(w, z)$, consider the following square:

$$\begin{array}{ccc} \mathcal{C}(z,x) & \xrightarrow{-\circ g} & \mathcal{C}(w,x) \\ \downarrow{\scriptstyle f\circ -} & & \downarrow{\scriptstyle f\circ -} \\ \mathcal{C}(z,y) & \xrightarrow{-\circ g} & \mathcal{C}(w,y) \end{array}$$

and for any $h \in \mathcal{C}(z, x)$, we can chase the diagram: going right, then down, yields $f \circ (h \circ g)$; going down and then right yields $(f \circ h) \circ g$. But associativity of composition forces these to be the same.

Now we prove that $\mathsf{Nat}(h^x, F) \cong F(x)$. This is the fundamental trick of the Yoneda lemma: understanding natural transformations corresponding to representable functors amounts to examining what occurs at the identity. Let $\eta : h^x \to F$ be a natural transformation. Then for any $y \in \mathcal{C}$, $f \in \mathcal{C}(y, x)$, we have the following diagram:

$$\begin{array}{ccc} \mathcal{C}(x,x) & \xrightarrow{-\circ f} & \mathcal{C}(y,x) \\ \downarrow{\scriptstyle \eta_x} & & \downarrow{\scriptstyle \eta_y} \\ F(x) & \xrightarrow{F(f)} & F(y) \end{array}$$

and applying commutativity to the element $\mathrm{id}_x \in \mathcal{C}(x, x)$, we see that $\eta_y(f) = F(f)(\eta_x(\mathrm{id}_x))$. This means that $\eta \in \mathsf{Nat}(h^x, F)$ is uniquely determined by its value at x. Conversely, let $a \in F(x)$. Define the natural transformation $\eta^a : h^x \to F$ by $\eta^a_y(f) = F(f)(a)$. Again, this is a natural transformation: if $g \in \mathcal{C}(y, z)$, we chase the diagram

$$\begin{array}{ccc} \mathcal{C}(z,x) & \xrightarrow{-\circ g} & \mathcal{C}(y,x) \\ \downarrow{\scriptstyle F(-)(a)} & & \downarrow{\scriptstyle F(-)(a)} \\ F(z) & \xrightarrow{F(g)} & F(y) \end{array}$$

Pick a morphism $f \in \mathcal{C}(z, x)$. Going right, then down sends f to $F(f \circ g)(a)$. Going down, then right sends f to

$$F(g)(F(f)(a)) = F(g)F(f)(a) = F(f \circ g)(a).$$

Because these processes are obviously mutually inverse, this establishes the isomorphism. Now that we know this is an isomorphism, we can establish naturality in $\mathcal{C} \times \mathsf{PSh}(\mathcal{C})$. First, let $\zeta : F \to G$ be a natural transformation of presheaves. The isomorphism $\mathsf{Nat}(h^x, F) \cong F(x)$ is established by taking a natural transformation η and sending it to $\eta_x(\mathrm{id}_x)$. So consider the following commutative diagram of sets ($\mathsf{Nat}(h^x, F)$ is a set because it is isomorphic to $F(x)$):

$$\begin{array}{ccc} \mathsf{Nat}(h^x, F) & \xrightarrow{\eta \mapsto \eta_x(\mathrm{id}_x)} & F(x) \\ \downarrow{\scriptstyle \eta \mapsto \zeta \circ \eta} & & \downarrow{\scriptstyle \zeta_x} \\ \mathsf{Nat}(h^x, G) & \xrightarrow{\xi \mapsto \xi_x(\mathrm{id}_x)} & G(x) \end{array}$$

Fix an $\eta \in \mathsf{Nat}(h^x, F)$. Going right, then down gives

$$\zeta_x \circ \eta_x(\mathrm{id}_x)$$

whereas going down, then right gives

$$(\zeta \circ \eta)_x(\mathrm{id}_x)$$

which are manifestly the same.

Similarly, to check naturality in $\mathcal{C}$, let $f \in \mathcal{C}(x, y)$. Then f induces a morphism $h^f : h^x \to h^y$, which in turn induces a function $\mathsf{Nat}(h^y, F) \to \mathsf{Nat}(h^x, F)$ for every $F \in \mathsf{PSh}(\mathcal{C})$. Now we chase the diagram:

$$\begin{array}{ccc} \mathsf{Nat}(h^y, F) & \xrightarrow{\eta \mapsto \eta_y(\mathrm{id}_y)} & F(y) \\ \downarrow{\scriptstyle \eta \mapsto h^f \circ \eta} & & \downarrow{\scriptstyle F(f)} \\ \mathsf{Nat}(h^x, F) & \xrightarrow{\xi \mapsto \xi_x(\mathrm{id}_x)} & F(x) \end{array}$$

and again we unpack definitions: because η is a natural transformation from h^y to F, $F(f) \circ \eta_y(\mathrm{id}_y) = \eta_x(f)$. Now, first going right, then down,

$$\begin{aligned} F(f)(\eta_y(\mathrm{id}_y)) &= \eta_x(f) \\ &= h^f_x(\eta_x)(\mathrm{id}_x) \\ &= (h^f \circ \eta)_x(\mathrm{id}_x) \end{aligned}$$

which shows the naturality in $\mathcal{C}$. □

Corollary 3.6. Representable functors are isomorphic if and only if the objects they are represented by are isomorphic.

Corollary 3.7. Let G be any group, considered as a one-object category where every morphism is an element of the group and composition is group multiplication. Applying the Yoneda lemma shows that G has a faithful action on some set (in this case, this is the left action on itself).

Remark 3.8. Various versions of an enriched Yoneda lemma are available. Here is one about Banach spaces: Yoneda for Banach-enriched categories.

Corollary 3.9. Applying Yoneda as above to $\mathcal{C}^{op}$ gives a contravariant embedding of $\mathcal{C}$ to the covariant functors $[\mathcal{C}, \mathsf{Set}]$. We denote the functor $\mathcal{C}(x, -)$ as ${}^{x}h$.

Corollary 3.10. Because Set is closed under all small limits and colimits, so is the category $\mathsf{PSh}(\mathcal{C})$, by taking limits and colimits in the target category.

Theorem 3.11. The Yoneda embedding is continuous (that is, it preserves limits).

Proof. Let $F : \mathcal{I} \to \mathcal{C}$ be a functor from a diagram category with limit $(\lim F, \epsilon_i)$. Let $G \in \mathsf{PSh}(\mathcal{C})$, and let $(\eta_i) : \Delta G \to \mathcal{C}(-, F(i))$ be a natural transformation of functors in the category $[\mathcal{I}, \mathsf{Set}]$: that is, for each $i \in \mathcal{I}$, a natural transformation $G \to \mathcal{C}(-, F(i))$ such that the following diagram commutes for each $f \in \mathcal{I}(i, i')$:

$$\begin{array}{ccc} G & & \\ \downarrow{\scriptstyle \eta_i} & \searrow^{\eta_{i'}} & \\ \mathcal{C}(-, F(i)) & \xrightarrow[Ff\circ -]{} & \mathcal{C}(-, F(i')) \end{array}$$

Now, for each $x \in \mathcal{C}$, we have a family of maps $(\eta_i)_x : Gx \to \mathcal{C}(x, F(i))$. For each element g of Gx, $(\eta_i)_x(g) \in \mathcal{C}(x, F(i))$ defines a map $\Delta x \to F$. So there is a unique lift $\lim_i(\eta_i)_x(g) : x \to \lim F$. This means that elements of Gx give elements of $\mathcal{C}(x, \lim F)$, and this is what is commonly known as a function. Call this function Φ_x.

Now it remains to show that this is natural in x; let $f \in \mathcal{C}(x, x')$, and $g \in G(x')$. $\Phi_x \circ Gf(g)$ is the unique morphism $\phi : x \to \lim F$ such that $\epsilon_i \circ \phi = (\eta_i)_x(Gf(g))$, by definition. But $[\Phi_{x'}(g)] \circ f$ also shares this property: $\epsilon_i[\Phi_{x'}(g)] \circ f = (\eta_i)_{x'}(g) \circ f = (\eta_i)_x(Gf(g))$, because η_i is a family of natural transformations. □

Corollary 3.12. Given a compact smooth manifold M, M can be recovered functorially from its ring of smooth functions $A = C^\infty(M)$.[1]

Proof. The points of M are in correspondence with maximal ideals of $C^\infty(M)$. Because maximal ideals is a functor on commutative algebras, we have recovered the set; we just need to recover the smooth structure. We already know that M is a manifold; we just have to recover which. But for any other compact manifold N, N embeds into some large $\mathbb{R}^n$ via an embedding i, and map $\phi : M \to N$ is smooth if and only if $i \circ \phi$ has smooth components. But we know all of smooth functions $M \to \mathbb{R}$: we have access to $C^\infty(M)$. So we have recovered the functor $\mathcal{C}(M, -)$, and, by opposite Yoneda, recovered M up to smooth isomorphism (diffeomorphism). $\square$

Corollary 3.13 (Monos and epis)**.** We can return to characterizing monomorphisms and epimorphisms as "injective" and "surjective" using the Yoneda lemma. A morphism $f \in \mathcal{C}(x, y)$ is a monomorphism if and only if for every $z \in \mathcal{C}$, the Yoneda natural transformation $(h^f)_z : \mathcal{C}(z, x) \to \mathcal{C}(z, y)$ is injective. Similarly, f is an epimorphism if and only if for every $z \in \mathcal{C}$, the opposite Yoneda natural transformation ${}^f h : \mathcal{C}(y, z) \to \mathcal{C}(x, z)$ is an injection.

Corollary 3.14 (Sections and retractions)**.** Similarly, we can characterize sections and retractions. A morphism $f \in \mathcal{C}(x, y)$ is a section if ${}^f h : \mathcal{C}(y, -) \to \mathcal{C}(x, -)$ is surjective for every $z \in \mathcal{C}$; a morphism $g \in \mathcal{C}(y, x)$ is a retraction if $h^g : \mathcal{C}(-, y) \to \mathcal{C}(-, x)$ is surjective for every $z \in \mathcal{C}$.

The following proof is long and technical, and can probably be skipped. Nevertheless, if you are interested, this is a good theorem to treat as an exercise. Given just the definition of $\mathcal{I}$, can you prove that F is the colimit of a functor coming from $\mathcal{I}$? What if you are given the functor?

Theorem 3.15 (Density theorem)**.** [3, 3.7 Theorem 1] Every functor in $\mathsf{PSh}(\mathcal{C})$ is the colimit of representable functors h^x. In this sense, representable functors are dense in $\mathsf{PSh}(\mathcal{C})$.

Proof. Throughout this proof, I dispense with our usual convention of using lowercase letters to represent elements of a category, and instead choose uppercase letters to represent them. Lowercase letters are reserved for elements of sets and morphisms.

First, fix a presheaf F. Define the indexing category $\mathcal{I}$:

1. The objects of $\mathcal{I}$ are pairs (X, a), with $X \in \mathcal{C}$, $a \in F(X)$.

2. The morphisms of $\mathcal{I}$ are pointed morphisms: if (X, a), (Y, b) are objects of $\mathcal{I}$, the morphisms from (X, a) to (Y, b) are those morphisms $f \in \mathcal{C}(X, Y)$ such that $Ff(b) = a$.

Let p denote the forgetful functor $\mathcal{I} \to \mathcal{C}$. Then F is the colimit of the functor $h \circ p : \mathcal{I} \to \mathsf{PSh}(\mathcal{C})$.

Define the structure morphisms $\eta_{(X,a)} : \mathcal{C}(-, X) \to F$ by $\eta_{(X,a)}(f) = Ff(a)$. These are natural transformations: for any $Y, Z \in \mathcal{C}$, $g \in \mathcal{C}(Y, Z)$, we have the following diagram

$$
\begin{array}{ccc}
\mathcal{C}(Z,X) & \xrightarrow{-\circ g} & \mathcal{C}(Y,X) \\
\downarrow{\scriptstyle f\mapsto Ff(a)} & & \downarrow{\scriptstyle f\mapsto Ff(a)} \\
F(Z) & \xrightarrow{F(g)} & F(Y)
\end{array}
$$

and chasing f around this diagram gives $F(f \circ g)(a) = F(g) \circ F(f)(a)$. Moreover, these are natural transformations with respect to the category $\mathcal{I}$: if $f \in \mathcal{I}((X,a),(Y,b))$, then we have the following commuting triangle

$$
\begin{array}{ccc}
\mathcal{C}(-,X) & & \\
 & \searrow^{g\mapsto F(g)(a)} & \\
\downarrow{\scriptstyle f\circ -} & & F \\
 & \nearrow_{g\mapsto F(g)(b)} & \\
\mathcal{C}(-,Y) & &
\end{array}
$$

with commutativity coming because, for any g in $\mathcal{C}(-,X)$, we have

$$
\begin{aligned}
F(f \circ g)(b) &= F(g) \circ F(f)(b) \\
&= F(g)(a)
\end{aligned}
$$

as desired.

Now we need to show that these morphisms are universal. Let

$$\phi_{(X,a)} : \mathcal{C}(-,X) \to G$$

be a family of natural transformations of functors commuting with the induced maps. Then for each $Z \in \mathcal{C}$, define the function $\phi_Z : F(Z) \to G(Z)$ by

$$\phi_Z(a) = (\phi_{(Z,a)})_Z(\mathrm{id}_Z).$$

To show that ϕ_Z defined in this way assembles to a natural transformation in the functor category, let $g \in \mathcal{C}(Z',Z)$, and examine the following diagram:

$$
\begin{array}{ccc}
F(Z) & \xrightarrow{Fg} & F(Z') \\
\downarrow{\scriptstyle \phi_Z} & & \downarrow{\scriptstyle \phi_{Z'}} \\
G(Z) & \xrightarrow{Gg} & G(Z')
\end{array}
$$

and fix $a \in F(Z)$. Then $g \in \mathcal{I}((Z', Fg(a)),(Z,a))$, and chasing a around we see that down, then right is

$$Gg(\phi_{Z,a})_Z(\mathrm{id}_Z)$$

whereas right, then down is

$$(\phi_{(Z',Fg(a))})_{Z'}(\mathrm{id}_{Z'}).$$

By the commutativity of

$$\begin{array}{ccc} \mathcal{C}(Z,Z) & \xrightarrow{(\phi_{(Z,a)})_Z} & G(Z) \\ \downarrow{\scriptstyle -\circ g} & & \downarrow{\scriptstyle Gg} \\ \mathcal{C}(Z',Z) & \xrightarrow{(\phi_{(Z,a)})_{Z'}} & G(Z') \end{array}$$

we see that

$$\begin{aligned} Gg(\phi_{(Z,a)})_Z(\mathrm{id}_Z) &= (\phi_{(Z,a)})_{Z'}(\mathrm{id}_{Z'} \circ g) \\ &= (\phi_{(Z,a)})_{Z'}(g) \end{aligned}$$

and by commutativity of

$$\begin{tikzcd} \mathcal{C}(Z',Z) \arrow[dr, "(\phi_{(Z,a)})_{Z'}"] & \\ & G(Z') \\ \mathcal{C}(Z',Z') \arrow[uu, "g\circ -"] \arrow[ur, "(\phi_{(Z',Fg(a))})_{Z'}"'] & \end{tikzcd}$$

we have

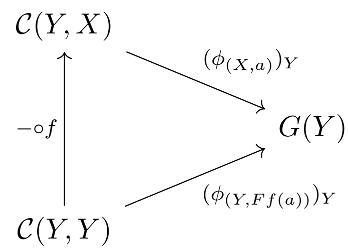
$$(\phi_{(Z',Fg(a))})_{Z'}(\mathrm{id}_{Z'}) = (\phi_{(Z,a)})_{Z'}(g)$$

which proves that ϕ assembles to a natural transformation in the functor category.

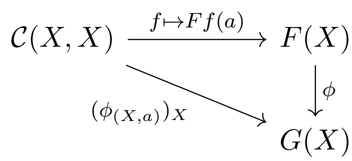
To show that $\phi \circ \eta_{(X,a)} = \phi_{(X,a)}$ for each $(X,a) \in \mathcal{I}$, we want to show that for every $Y \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}(Y,X) \arrow[r, "f\mapsto Ff(a)"] \arrow[dr, "(\phi_{(X,a)})_Y"'] & F(Y) \arrow[d, "\phi_Y"] \\ & G(Y) \end{tikzcd}$$

In order to do this, fix $f \in \mathcal{C}(Y,X)$. Then $\phi_Y(Ff(a)) = (\phi_{(Y,Ff(a))})_Y(\mathrm{id}_Y)$. To compute this morphism, consider the commuting triangle



so $\phi_Y(Ff(a)) = (\phi_{(X,a)})_Y(f)$ as desired. This means that ϕ is a morphism making the diagram commute. Uniqueness is, thankfully, obvious: for any $X \in \mathcal{C}$, $a \in F(X)$, for a morphism $\phi : F \to G$ to commute with the $\phi_{(X,a)}$ for all $(X,a) \in \mathcal{I}$, we must have the following diagram



and applying this to $f = \mathrm{id}_X$, we see that $\phi(a) = (\phi_{(X,a)})_X(\mathrm{id}_X)$, as defined above. $\square$

Corollary 3.16. The embedding $h : \mathcal{C} \to \mathsf{PSh}(X)$ is the free cocompletion of $\mathcal{C}$: if $F : \mathcal{C} \to \mathcal{D}$ is any functor to a cocomplete category $\mathcal{D}$, there is a unique cocontinuous functor $\hat{F} : \mathsf{PSh}(X) \to \mathcal{D}$ which extends F. This is in fact a 2-categorical adjunction from the category of locally small categories with functors and natural transformations to the category of locally small cocomplete categories with cocontinuous functors and natural transformations.

Remark 3.17. The moral of the Yoneda lemma, if there is one, is that whenever you are dealing with natural transformations of functors with a representable functor involved, try to see what's happening at the identity. That should force the behavior on on the rest of the category.

3.1 Exercises

1. Let $\mathcal{C}$ be the category of G-sets, and U the forgetful functor $\mathcal{C} \to \mathsf{Set}$. Show that $[\mathcal{C}, \mathsf{Set}](U,U) \cong G$ as a group.

   [Hint: $U \cong \mathcal{C}((G, \text{left translation}), -)$. Apply Yoneda.]

2. Let $\mathcal{C}$ be the category of left R-modules, and let $U : \mathcal{C} \to \mathsf{Ab}$ be the forgetful functor to the category of Abelian groups. Using whatever version of enriched Yoneda you feel comfortable with, show

$$[\mathcal{C}, \mathsf{Ab}](U, U) \cong R.$$

3. Find an example of two functors $F, G : \mathcal{C} \to \mathcal{D}$, with, for each $x \in \mathcal{C}$, $F(x) \cong G(x)$, but $F \ncong G$. Find representable functors with the same property.

4. Find an example of a full and faithful functor which fails to be an isomorphism of categories.

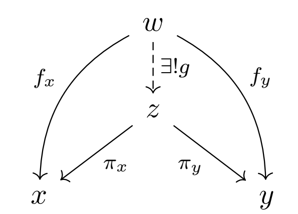
4 Limits and Colimits

Limits and colimits can be thought of as generalized ways of gluing together objects to make new objects.

4.1 Products and Coproducts

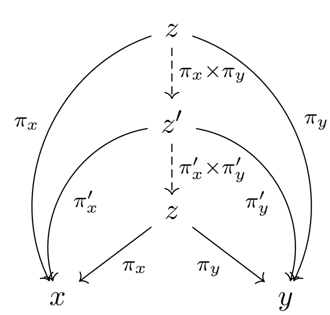
Fix a category $\mathcal{C}$.

Definition 4.1. Given objects $x, y \in \mathcal{C}$, an object $z \in \mathcal{C}$ together with morphisms $\pi_x : z \to x, \pi_y : z \to y$ is called the **product** of x and y if it satisfies the following universal property: for every $w \in \mathcal{C}$, $f_x \in \mathcal{C}(w, x)$, $f_y \in \mathcal{C}(w, y)$, there exists a unique morphism $f_x \times f_y = g \in \mathcal{C}(w, z)$ that makes the following diagram commute:



Lemma 4.2. If (z, π_x, π_y) and (z', π'_x, π'_y) are both the product of x and y, then they are isomorphic with a unique isomorphism preserving the projections.

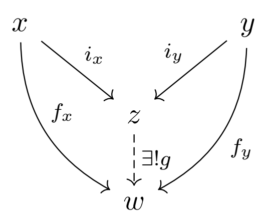
Proof. Examine the following diagram:



and note that $(\pi'_x \times \pi'_y) \circ (\pi_x \times \pi_y) : z \to z$ is a morphism which commutes with the projections π_x and π_y by commutativity of the diagram. By the product property of (z, π_1, π_2), we know exactly one morphism does that, but the identity obviously satisfies this property. So $(\pi'_x \times \pi'_y) \circ (\pi_x \times \pi_y) = \mathrm{id}_z$. Similarly, the other composition is $\mathrm{id}_{z'}$. So the objects are in canonical isomorphism.

We could also have proved this using the Yoneda lemma, which we'll do later. □

Definition 4.3. Given $x, y \in \mathcal{C}$, an object z together with maps $i_x \in \mathcal{C}(x, z)$, $i_y \in \mathcal{C}(y, z)$ is their **coproduct** if for every $w \in \mathcal{C}$ and maps $f_x \in \mathcal{C}(x, w), f_y \in \mathcal{C}(y, w)$, there is a unique morphism $f_x \coprod f_y = g \in \mathcal{C}(z, w)$ making the following diagram commute:



Lemma 4.4. Coproducts are unique up to unique isomorphism.

Proof. Exercise. □

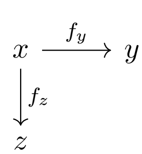
Example 4.5. In the category Top, the product of two topological spaces is their setwise product with the product topology; the coproduct is their disjoint union.

Example 4.6. In the category of left R-modules, the product and the coproduct of two modules M, N are both given by $M \oplus N$. In fact, in any additive category, finite products and coproducts coincide.

Example 4.7. In the category of pointed topological spaces $\top_*$, the product is the smash product: $(X, x_0) \times (Y, y_0) \cong (X \times Y/(x_0, y) \sim (x, y_0), [(x_0, y_0)])$. The coproduct is the wedge sum $(X \coprod Y)/(x_0 = y_0)$.

Example 4.8. The coproduct of k-algebras is given by their tensor product (over k). Their product is actually just their setwise product endowed with the standard operations.

Definition 4.9. Given three objects $x, y, z \in \mathcal{C}$, and morphisms $f_y \in \mathcal{C}(x, y), f_z \in \mathcal{C}(x, z)$, we say that an object $p \in \mathcal{C}$ together with maps $i_y \in \mathcal{C}(y, p), i_z \in \mathcal{C}(z, p)$ such that $i_y \circ f_y = i_z \circ f_z$ is the **pushout** of the diagram



if, for every object $q \in \mathcal{C}$ and maps $\phi_y \in \mathcal{C}(y, q), \phi_z \in \mathcal{C}(z, q)$ with $\phi_y \circ f_y = \phi_z \circ f_z$, there exists a unique morphism $\phi \in \mathcal{C}(p, q)$ making the following diagram commute:

$$\begin{tikzcd} x \arrow[r, "f_y"] \arrow[d, "f_z"] & y \arrow[d, "i_y"] \arrow[ddr, bend left, "\phi_y"] & \\ z \arrow[r, "i_z"] \arrow[drr, bend right, "\phi_z"'] & p \arrow[dr, dashed, "\exists!\phi"] & \\ & & q \end{tikzcd}$$

Note that when x is the initial object in the category, p is automatically the coproduct of y and z. So p is some object resembling a coproduct. (In fact, it is the coproduct in the coslice category $c \downarrow \mathcal{C}$.)

Example 4.10. The category of commutative k-algebras is the coslice category under k of commutative rings; the coproduct in this category is the pushout of the diagram

$$\begin{array}{ccc} k & \xrightarrow{\eta_1} & A_1 \\ \downarrow{\scriptstyle \eta_2} & & \\ A_2 & & \end{array}$$

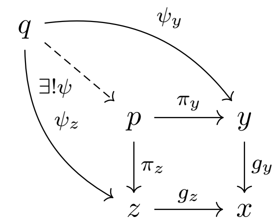
Example 4.11. The category of pointed topological spaces Top_* is the coslice category under a point; the coproduct is now the pushout of the diagram

$$\begin{array}{ccc} \{*\} & \xrightarrow{x_0} & X \\ \downarrow{\scriptstyle y_0} & & \\ Y & & \end{array}$$

Definition 4.12. Similarly, the pullback of three objects $x, y, z \in \mathcal{C}$, together with morphisms $g_y \in \mathcal{C}(y, x), g_z \in \mathcal{C}(z, x)$

$$\begin{array}{ccc} & & y \\ & & \downarrow{\scriptstyle g_y} \\ z & \xrightarrow{g_z} & x \end{array}$$

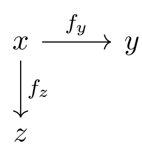
is an object $p \in \mathcal{C}$, together with morphisms $\pi_y : p \to y, \pi_z : p \to z$, such that $g_y \circ \pi_y = g_z \circ \pi_z$, such that for every object q with $\psi_y \in \mathcal{C}(q, y)$, $\psi_z \in \mathcal{C}(q, z)$ with $g_y \circ \psi_y = g_z \circ \psi_z$, there is a unique $\psi \in \mathcal{C}(q, p)$ making the diagrams commute:



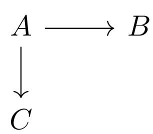
Again, note that taking x to be a trivial object – here, the terminal object of the category – makes p the product in a canonical way.

It should be clear now that we should be able to take "products" and "coproducts" over arbitrary diagrams in $\mathcal{C}$. But what is a diagram? A diagram is a functor from a diagram category to $\mathcal{C}$.

A diagram with two objects and no morphisms between them is a functor from the category $\mathbb{2}$, which has the objects $\{1, 2\}$, and no nontrivial morphisms. A diagram such as



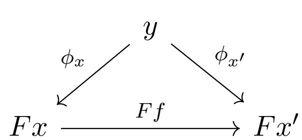
is the same as a functor from the category



Obviously, we can phrase more complicated diagrams, with more complicated commutativity conditions, as functors from categories with more complicated structure.

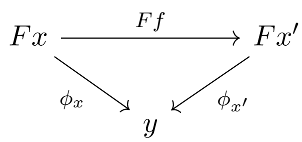
4.2 Cones and Cocones

Definition 4.13. Denote the indexing category by $\mathcal{I}$. If F is a functor $F \in [\mathcal{I}, \mathcal{C}]$, a **cone** over F is an object $y \in \mathcal{C}$ together with morphisms $\phi_x : y \to Fx$ for every $x \in \mathcal{I}$ such that the following diagram commutes for every $f \in \mathcal{I}(x, x')$:

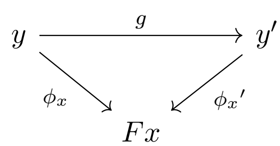


Equivalently, a cone is a morphism in the functor category $[\mathcal{I}, \mathcal{C}]$ from the constant functor Δy to F.

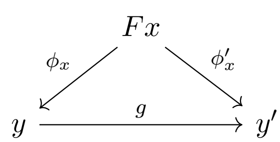
Definition 4.14. A cocone is the same thing, but with all the arrows reversed. A cocone under F is a morphism in $[\mathcal{I}, \mathcal{C}]$ from F to Δy for some y.



A morphism of cones or cocones is a morphism in the slice category over F, over a morphism in the coslice category under F. Explicitly, if (y, ϕ_x) and $(y', \phi'_{x'})$ are two cones over F, a morphism between them is a map $g \in \mathcal{C}(y, y')$ such that for every $x \in \mathcal{I}$,



and similarly, a map of cocones is a $g \in \mathcal{C}(y, y')$ such that for every $x \in \mathcal{I}$,



Definition 4.15. Given a functor F, a **limit** of F is a universal cone over F: a cone (z, ψ_x) such that for every other cone (y, ϕ_x), there is a unique morphism of cones from $(y, \phi_x) \to (z, \psi_x)$. In this case, we write (z, ψ_x) as $\lim F$.

This statement says that $\mathcal{C}(-, z) \cong [\mathcal{I}, \mathcal{C}](\Delta -, F)$. By Yoneda, this guarantees uniqueness of z up to unique isomorphism; the structure maps of z come for free from the isomorphism.

Definition 4.16. Given a functor F, a **colimit** of F is a universal cocone: a cocone (z, ψ_x) such that for every other cocone (y, ϕ_x), there is a unique morphism of cones $(z, \psi_x) \to (y, \phi_x)$. In this case, we write (z, ψ_x) as $\operatorname{colim} F$.

This says that $\mathcal{C}(z,-) \cong [\mathcal{I},\mathcal{C}](F,\Delta -)$. Again, uniqueness of the object comes from representability.

Example 4.17 (Trivial limits)**.** Let $\mathcal{I}$ be the category

$$1 \longrightarrow 2 \longrightarrow \dots$$

and F any functor $F : \mathcal{I} \to \mathcal{C}$. Then F always has a limit, and the limit is isomorphic to $F(1)$. This can intuitively be seen by the fact that a cone (y, ϕ_x) is uniquely determined by ϕ_1.

Example 4.18 (Trivial colimits)**.** Similarly, if $\mathcal{I}$ is the category

$$\dots \longrightarrow 2 \longrightarrow 1$$

and F is any functor $F : \mathcal{I} \to \mathcal{C}$, F has a colimit, with the object being $F(1)$. Again, we can see this from the fact that a cocone (y, ϕ_x) is uniquely determined by ϕ_1.

This indicates to us that limits are related to initial objects, and colimits are related to terminal objects.

Example 4.19. Objects are often colimits of distinguished subobjects. For example, every module is a colimit of finitely generated modules – its own submodules. Every manifold is a colimit of copies of $\mathbb{R}^n$.

Example 4.20. Fix a prime number p, and consider the system

$$\mathbb{Z}/p\mathbb{Z} \xrightarrow{1 \mapsto p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{1 \mapsto p} \mathbb{Z}/p^3\mathbb{Z} \to \dots$$

The limit over this diagram is just $\mathbb{Z}/p\mathbb{Z}$; the colimit, referred to as the Prüfer group $\mathbb{Z}(p^\infty)$, which can be thought of as the union of these groups.

Example 4.21. Fix a prime number p, and consider the system

$$\dots \to \mathbb{Z}/p^3\mathbb{Z} \xrightarrow{1 \mapsto 1} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{1 \mapsto 1} \mathbb{Z}/p\mathbb{Z}$$

The colimit of this diagram is just $\mathbb{Z}/p\mathbb{Z}$; the limit is the p-adic integers, which can be identified with sequences of residue classes $(a_0, a_1, \dots)$ with $p | (a_{i+1} - a_i)$ for all i.

Example 4.22. A CW complex is precisely a colimit of standard n-disks $\mathbb{D}^n$ in Top, with the only maps allowed being attachment maps. When we combine this with particular adjointness results about functors, this allows for easier computations with respect to CW complexes.

Definition 4.23. A category $\mathcal{C}$ is called **complete** if, for every small category $\mathcal{I}$ and functor $F : \mathcal{I} \to \mathcal{C}$, $\lim F$ exists. Similarly, $\mathcal{C}$ is called **cocomplete** if for every small category $\mathcal{I}$ and functor $F : \mathcal{I} \to \mathcal{C}$, $\operatorname{colim} F$ exists.

Remark 4.24. The general recipe for building a limit in concrete categories is as follows: the largest possible limit is the product over all of the objects of $\mathcal{I}$, $\Pi_{x \in \mathcal{I}} Fx$. Quotienting this product by appropriate relations gives an object which is necessarily the limit of the diagram.

Remark 4.25. Conversely, to build a colimit in a concrete category, we first take the maximal colimit: $\coprod_{x \in \mathcal{I}} Fx$. Then the colimit of the diagram F is the coherent sequences in $\coprod_{x \in \mathcal{I}} Fx$: the elements $(t_x)_{x \in \ I}$, with each $t_x \in Fx$, such that for each $f \in \mathcal{I}(x, x')$, $Ff(t_x) = t_{x'}$.

Because the presheaf category $\mathsf{PSh}(\mathcal{C})$ is complete, and the Yoneda embedding is continuous, we can always find a limit inside the presheaf category, and then check if that functor is representable.

5 Adjoint Functors

5.1 Introduction

Definition 5.1 (Adjoint functors)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that (F, G) are an **adjoint pair**, with F being the **left adjoint** and G being the **right adjoint**, if there is an isomorphism of the following two functors $C^{op} \times \mathcal{D} \to \mathsf{Set}$: $\mathcal{C}(-, G-)$ and $\mathcal{D}(F-, -)$. In this case, we write $F \dashv G$.

Let us make explicit the isomorphism that occurs here. For each $(x, y) \in \mathcal{C} \times \mathcal{D}$, there is an isomorphism $\Phi_{x,y} : \mathcal{C}(x, Gy) \to \mathcal{D}(Fx, y)$ satisfying the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(x, Gy) & \xrightarrow{\mathcal{C}(f,Gg)} & \mathcal{C}(x', Gy') \\ \downarrow{\scriptstyle \Phi_{x,y}} & & \downarrow{\scriptstyle \Phi_{x',y'}} \\ \mathcal{D}(Fx, y) & \xrightarrow{\mathcal{D}(Ff,g)} & \mathcal{D}(Fx', y') \end{array}$$

where the horizontal morphisms are defined in the following way: if $f \in C^{op}(x, x'), g \in \mathcal{D}(y, y')$, $\phi : x \to Gy$, $\mathcal{C}(f, Gg)$ is defined as the following composition: $Gg \circ \phi \circ f$. Similarly, if $\psi \in \mathcal{D}(Fx, y)$, $\mathcal{D}(Ff, g)$ is defined by $g \circ \psi \circ Ff$, as in the following diagrams.

$$\begin{array}{ccccc} x' & \xrightarrow{f} & x & & \\ & & \downarrow{\scriptstyle \phi} & & \\ & & Gy & \xrightarrow{Gg} & Gy' \end{array} \qquad \begin{array}{ccccc} Fx' & \xrightarrow{Ff} & Fx & & \\ & & \downarrow{\scriptstyle \psi} & & \\ & & y & \xrightarrow{g} & y' \end{array}$$

This is generally stated as being an isomorphism that is natural in both arguments.

Example 5.2 (The discrete space)**.** There is a functor $G : \mathsf{Top} \to \mathsf{Set}$, generally called the **forgetful functor**, which takes a topological space to its underlying set, and a continuous function to its underlying function. Define a functor $F : \mathsf{Set} \to \mathsf{Top}$ which takes a set X to the topological space $(X, 2^X)$, and a function $f : X \to X'$ to the same function $(X, 2^X) \to (X', 2^{X'})$. Note that f is actually a continuous map, because the preimage of an open set in $(X', 2^{X'})$ is open, because every set in $(X, 2^X)$ is open.

In fact, $F \dashv G$. Let's check this explicitly. First, define

$$\Phi_{x,y} : \mathsf{Set}(x, Gy) \to \mathsf{Top}(Fx, y)$$

by noting that every function from a set with the discrete topology is automatically continuous, and every continuous map arises from a particular function on the underlying sets. Let (Y, τ) and (Y', τ') be topological spaces $g : (Y, \tau) \to (Y', \tau')$ a continuous function, and X, X' two sets, $f : X' \to X$ a function. Now, let $\phi \in \mathsf{Set}(X, GY)$: that is, a function from X to the set Y. We wish to show that $\mathsf{Top}(Ff, g) \circ \Phi_{X,Y} \circ \phi = \Phi_{X',Y'} \circ \mathsf{Set}(f, Gg) \circ \phi$. We can write out the left-hand side:

$$\begin{aligned} \mathsf{Top}(Ff, g) \circ \Phi_{X,Y}\phi &= \mathsf{Top}(Ff, g)\phi \\ &= g \circ \phi \circ f \end{aligned}$$

inside the category of sets, where everything is living. On the other side,

$$\begin{aligned} \Phi_{X',Y'} \circ \mathsf{Set}(f, Gg) \circ \phi &= \Phi_{X',Y'}(g \circ \phi \circ f) \\ &= g \circ \phi \circ f \end{aligned}$$

so the diagram commutes and Φ actually defines a natural isomorphism.

Example 5.3 (Free functors)**.** Many categories have objects which are sets together with some additional structure, and morphisms between objects are functions between the sets that preserve the structure in some way. In this case, there is a **forgetful functor** $\mathcal{C} \to \mathsf{Set}$. Any left adjoint to a forgetful functor is called a **free functor**. Many categories, particularly algebraic categories, often enjoy free functors; to name a few, $\mathsf{Grp}, \mathsf{Alg}, R - \mathsf{mod}, R - \mathsf{Mod}$.

Example 5.4 (Hom-tensor adjunction)**.** One of the classical adjunctions is hom-tensor adjunction. In its simplest form, we can state it as follows: let R be a commutative ring, and $R - \mathsf{Mod}$ its category of modules. There is a functor $\otimes : R - \mathsf{Mod} \times R - \mathsf{Mod} \to R - \mathsf{Mod}$, and an internal functor $\mathrm{Hom}_R : R - \mathsf{Mod}^{op} \times R - \mathsf{Mod} \to R - \mathsf{Mod}$. Then for each each $B \in R - \mathsf{Mod}$, the functors $F = - \otimes B$ and $G = \mathrm{Hom}_R(B, -)$ satisfy $F \dashv G$. In fact, this isomorphism is much stronger and more general, but this should hold for now.

Remark 5.5. Proving the adjointness for just the free topological space was time-consuming. Below is an equivalent characterization of adjointness that is sometimes easier to prove, and is sometimes more useful for proofs.

Definition 5.6 (unit-counit definition of adjointness)**.** Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$. $F \dashv G$ if and only if there are two natural transformations $\eta : 1_{\mathcal{C}} \to GF$, $\epsilon : FG \to 1_{\mathcal{D}}$, such that the following coherence equations hold:

1. $F \to F1_{\mathcal{C}} \xrightarrow{1_F\eta} FGF \xrightarrow{\epsilon 1_F} 1_{\mathcal{D}}F \to F$ is 1_F;

2. $G \to 1_{\mathcal{D}}G \xrightarrow{\eta 1_G} GFG \xrightarrow{1_G\epsilon} G1_{\mathcal{D}} \to G$ is 1_G.

We call η the **unit** and ϵ the **counit** of the adjunction.

Remark 5.7. Satisfying the equations shown above is generally called **satisfying zizag**, after the graphical calculus for dualizable objects in a monoidal category.

Theorem 5.8 (equivalence of unit-counit and hom-set adjunction)**.** These two definitions are equivalent, and the data of the hom-set isomorphism and the unit and counit can be recovered from each other.

Proof. Assume that $F \dashv G$, $F : \mathcal{C} \to \mathcal{D}$. Define η by, for each $x \in \mathcal{C}$,

$$\eta_x = \Phi^{-1}_{x,Fx}(\mathrm{id}_{Fx})$$

and similarly, define ϵ by, for each $y \in \mathcal{D}$,

$$\epsilon_y = \Phi_{Gy,y}(\mathrm{id}_{Gy}).$$

To show that η is a natural transformation, we will use the fact that it comes from the Yoneda lemma. The functor $h^X = \mathcal{C}(-, x)$ has a natural transformation to $\mathcal{D}(F-, Fx)$ by the definition of a functor. $\Phi^{-1}_{-,Fx}$ provides a natural transformation to the functor h^{GFx}, so we have a natural transformation $\Phi^{-1}_{-,Fx} \circ h^F : h^x \to h^{GFx}$. Note that η is natural in x because all of the preceding natural transformations were natural in x as well. This morphism of representable functors is instantiated by the value at the identity, which gives us our definition of η, which is now manifestly natural. Similarly, ϵ can be constructed from the opposite Yoneda lemma, defined by letting ${}^yh = \mathcal{D}(y, -)$, applying G to obtain ${}^Gh \circ {}^yh$, and then applying $\Phi_{Gy,-}$ to land back in ${}^{FGy}h$. Because this version of Yoneda is contravariant, it actually induces a morphism $FGy \to y$.

To show that ϵ, η satisfy zigzag, we can again apply Yoneda. First, we need identities about composition of η, ϵ [2]. First, using the fact that Φ is natural in x, for any $f : x' \to x$, we have the following commuting square:

$$\begin{array}{ccc} \mathcal{C}(x, GFx) & \xrightarrow{\Phi_{x,Fx}} & \mathcal{D}(Fx, Fx) \\ \downarrow{\scriptstyle -\circ f} & & \downarrow{\scriptstyle -\circ Ff} \\ \mathcal{C}(x', GFx) & \xrightarrow{\Phi_{x',Fx}} & \mathcal{D}(Fx', Fx) \end{array}$$

and applying this to $\eta_x = \Phi^{-1}_{x,Fx}(\mathrm{id}_{Fx})$, we see that $\eta_x f = \Phi^{-1}_{x',Fx}(Ff)$. Similarly, if $f : Fx \to y$, we have the diagram

$$\begin{array}{ccc} \mathcal{C}(x, GFx) & \xrightarrow{\Phi x, Fx} & \mathcal{D}(Fx, Fx) \\ \Big\downarrow{\scriptstyle Gf\circ -} & & \Big\downarrow{\scriptstyle f\circ -} \\ \mathcal{C}(x, Gy) & \xrightarrow{\Phi_{x,y}} & \mathcal{D}(Fx, y) \end{array}$$

and applying this to η_x, we get that $Gf \circ \eta_x = \Phi^{-1}_{x,y}(f)$.

Similar calculations show that $\epsilon \circ Ff = \Phi(f)$, and $f \circ \epsilon = \Phi(Gf)$. Now we apply this to the zigzag equations:

1. At a particular object x,

$$\begin{aligned} \epsilon_{Fx} \circ F(\eta_x) &= \Phi(\eta_x) \\ &= \mathrm{id}_{Fx}. \end{aligned}$$

2. At a particular object y,

$$\begin{aligned} G(\epsilon_y) \circ \eta_{Gy} &= \Phi^{-1}(\epsilon_y) \\ &= \mathrm{id}_{Gy}. \end{aligned}$$

Now assume that $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ possess natural transformations $\eta : 1_{\mathcal{C}} \to GF$, $\epsilon : FG \to 1_{\mathcal{D}}$ satisfying zigzag. Then for each $x \in \mathcal{C}, y \in \mathcal{D}$, define a natural isomorphism $\Phi : \mathcal{C}(x, Gy) \to \mathcal{D}(Fx, y)$ by

$$\Phi_{x,y}(f) = \epsilon_y \circ Ff.$$

Naturality of ϵ, η implies that if $g \in \mathcal{D}(y, y')$, then $g \circ \epsilon_y = \epsilon_{y'} \circ FGg$, and if $f \in \mathcal{C}(x, x')$, then $GFf \circ \eta_x = \eta_{x'} \circ f$. Now we can find an explicit inverse to Φ, defined by $g \mapsto Gg \circ \eta_x$. Compute:

$$\begin{aligned} G(\epsilon_y \circ Ff) \circ \eta_x &= G(\epsilon_y) \circ GFf \circ \eta_x \\ &= G(\epsilon_y) \circ \eta_x \circ f \\ &= f \end{aligned}$$

and

$$\begin{aligned} \epsilon_y \circ F(Gg \circ \eta_x) &= \epsilon_y \circ FGg \circ F\eta_x \\ &= g \circ \epsilon_y \circ F\eta_x \\ &= g \end{aligned}$$

so composition both ways is the identity. To check naturality, let $f : x' \to x$, $g : y \to y'$. Then we have the following square:

$$\begin{array}{ccc} C(x, Gy) & \xrightarrow{\epsilon_y \circ F-} & \mathcal{D}(Fx, y) \\ \big\downarrow{\scriptstyle Gg\circ - \circ f} & & \big\downarrow{\scriptstyle g\circ - \circ Ff} \\ \mathcal{C}(x', Gy') & \xrightarrow{\epsilon_{y'} \circ F-} & \mathcal{D}(Fx', y') \end{array}$$

and going right, then down has the expression

$$g \circ \epsilon_y \circ F - \circ Ff$$

whereas going down, then right is

$$\epsilon_{y'} \circ F(Gg \circ - \circ f).$$

But both are equivalent by the fact that F is a functor and $\epsilon_{y'} \circ FGg = g \circ \epsilon_y$.

To recover the counit and unit once they have induced the hom-set isomorphism, if we define $\Phi_{x,y}(f) = \epsilon_y \circ Ff$, then $\Phi_{Gy,y}(\mathrm{id}_{Gy}) = \epsilon_y \circ \mathrm{id}_{FGy} = \epsilon_y$; similarly, $\Phi^{-1}_{x,Fx}(\mathrm{id}_{Fx}) = G(\mathrm{id}_{Fx}) \circ \eta_x = \mathrm{id}_{GFx} \circ \eta_x = \eta_x$.

Conversely, assume that Φ has given unit η and counit ϵ as above. Then

$$\begin{aligned} \epsilon_y \circ Ff &= \Phi_{Gy,y}(\mathrm{id}_{Gy}) \circ Ff \\ &= \Phi_{x,y}(f) \end{aligned}$$

by naturality of Φ. □

Example 5.9 (Units and counits)**.** In the world of free functors, we can think of the unit as being the "natural inclusion" from the original set to the new set that is built with the extra structure. For example, when taking the free group on a set X, FX, the unit is defined by taking each element of X to the letter it represents. The counit maps from the free group on a group back to the original group; it is the presentation of the group where each word is a generator.

In Top, the unit is actually the identity: it simply takes a set to itself via the identity morphism. However, the counit is a continuous map from the space equipped with the discrete topology to itself with its original topology.

Example 5.10 (Unit and counit for tensor adjunction)**.** Let $\mathcal{C}$ be the category $\mathbb{Z} - \mathsf{Mod}$, also denoted Ab, and consider $F \dashv G$, where F is the functor $- \otimes \mathbb{Z}_2$, and G is $\mathrm{Hom}(\mathbb{Z}_2, -)$. Then η_M is the map that takes $m \in M$ to the linear map $a \mapsto a \otimes m$, and ϵ_M is the map that takes an element of $\mathrm{Hom}(\mathbb{Z}_2, M) \otimes \mathbb{Z}_2$, say, $\sum \phi_i \otimes a_i$, and evaluates: $\sum \phi_i(a_i)$. (Note that these formulae work for any abelian group).

We can also define an adjunction by the following universal property: F and G are adjoint if, for every $y \in \mathcal{D}$, there is a map $\epsilon_y : FGy \to y$ such that, for every $g : Fx \to y$, there exists a unique $f : x \to Gy$ such that $g = \epsilon_y \circ Ff$. The data of η can be recovered similarly; this is just hom-set adjunction in disguise.

Theorem 5.11 (Uniqueness of adjoints)**.** Let $F \dashv G, F \dashv G'$. Then $G \cong G'$.

Proof. This is an easy exercise in unit/counit messing about. Let $\eta : 1_C \to GF, \eta' : 1_C \to G'F$ be the respective units; $\epsilon : FG \to 1_{\mathcal{D}}, \epsilon' : FG \to 1_{\mathcal{D}}$ their counits. By Yoneda, we need to exhibit a natural isomorphism $\mathcal{C}(-, G-) \to \mathcal{C}(-, G'-)$. Define this transformation by means of the following diagram:

$$\begin{array}{ccc} \mathcal{C}(x, Gy) & \xrightarrow{\epsilon_y \circ F-} & \mathcal{D}(Fx, y) \\ & & \downarrow \mathrm{id}_{\mathcal{D}} \\ \mathcal{C}(x, G'y) & \xleftarrow[G-\circ \eta'_x]{} & \mathcal{D}(Fx, y) \end{array}$$

which is explicitly defined by the component at y

$$G(\epsilon_y \circ F-) \circ \eta'_x$$

with inverse

$$G(\epsilon'_y \circ F-) \circ \eta_x.$$

We can explicitly check inversion:

$$\begin{aligned} G(\epsilon \circ F(G(\epsilon' \circ F-) \circ \eta)) \circ \eta' &= G(\epsilon \circ FG\epsilon' \circ FGF - \circ F\eta) \circ \eta' \\ &= G(\epsilon' \circ \epsilon \circ FGF - \circ F\eta) \circ \eta' \\ &= G(\epsilon' \circ F - \circ \epsilon \circ F\eta) \circ \eta' \\ &= G\epsilon' \circ GF - \circ \eta' \\ &= G\epsilon' \circ \eta' \circ - \\ &= - \end{aligned}$$

and symmetry implies that this is a two-sided inverse.

The exact same proof shows that left adjoints are unique up to canonical isomorphism. □

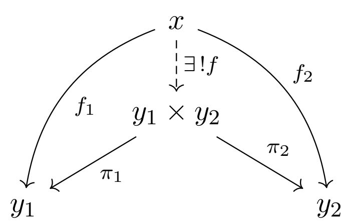
5.2 Limits and Colimits

Limits and colimits are particular instances of adjoint functors, when they are defined for an entire category of functors. Let us start with the simplest example.

Example 5.12 (Products and coproducts)**.** Let $\mathbb{2}$ denote the category with two objects, $\{1, 2\}$, and no non-identity morphisms, and let $\mathcal{C}^{\mathbb{2}}$ denote the category of functors $\mathsf{Fun}(\mathbb{2}, \mathcal{C})$. Objects in this category are indexed pairs of objects in $\mathcal{C}$; morphisms are indexed pairs of morphisms in $\mathcal{C}$. There is a functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathbb{2}}$ defined as follows: For each $x \in \mathcal{C}$, $\Delta(x)(i) = x$ for each $i \in \{1, 2\}$. For each $f : x \to x'$, $\Delta(f)(i) = f$.

$$\begin{array}{ccccc} x & & x & & x \\ \Big\downarrow{\scriptstyle f} & \Rightarrow & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f} \\ x' & & x' & & x' \end{array}$$

Let's first examine the functor $\mathcal{C}^{\mathbb{2}}(\Delta-, -)$. Specifically, $\mathcal{C}^{\mathbb{2}}(\Delta x, \{y_1, y_2\})$ is the collection of pairs of morphisms $f_1 \in \mathcal{C}(x, y_1), f_2 \in \mathcal{C}(x, y_2)$. Recall the diagram for the product:



That is, $\mathcal{C}(x, y_1 \times y_2) \cong \mathcal{C}^{\mathbb{2}}(\Delta x, \{y_1, y_2\})$, with the isomorphism being instantiated by the counit: $f \mapsto \{\pi_1 \circ f, \pi_2 \circ f\}$. In fact, the product functor is right adjoint to the diagonal embedding. The unit is what we think of as the standard diagonal embedding $x \to x \times x$, and the counit is the natural transformation

$$\begin{array}{ccc} y_1 \times y_2 & & y_1 \times y_2 \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_2} \\ y_1 & & y_2 \end{array}$$

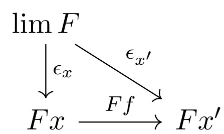
Similarly, the coproduct is a left adjoint to Δ. The unit is now in the functor category; it is the natural transformation

$$\begin{array}{ccc} x_1 & & x_2 \\ \big\downarrow{\scriptstyle i_1} & & \big\downarrow{\scriptstyle i_2} \\ x_1 \amalg x_2 & & x_1 \amalg x_2 \end{array}$$

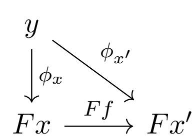
whereas the counit goes from $y \amalg y \to y$, and is generally called the **fold** map, ∇.

Example 5.13 (Limits over diagrams)**.** Recall that if $\mathcal{C}, \mathcal{D}$ are categories, $F : \mathcal{C} \to \mathcal{D}$, a limit over F is an object $\lim F \in \mathcal{D}$ together with morphisms $\epsilon_x : \lim F \to Fx$ for every $x \in C$ such that the following properties hold:

1. For every $f \in \mathcal{C}(x, x')$, the following diagram commutes:



2. For every $y \in \mathcal{D}$ and family of maps $\phi_x : y \to Fx$ such that the following diagram commutes for every $f \in \mathcal{C}(x, x')$



   there is a unique morphism $\lim \phi : y \to \lim F$ such that $\phi_x = \epsilon_x \lim \phi$ for every $x \in \mathcal{C}$.

This is precisely the definition of a right adjoint! Let us set up the machinery. We have a functor $\Delta : \mathcal{D} \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ which takes any object y to the functor $F(x) = y, F(f) = \mathrm{id}_y$, and takes any $f : y \to y'$ to the natural transformation ζ with component $\zeta_x = f$. Saying that limits over diagrams in $\mathcal{C}$ exist in $\mathcal{D}$ is exactly the same as saying that Δ possesses a right adjoint. In the same vein, saying that colimits over $\mathcal{C}$-diagrams exist in $\mathcal{D}$ is saying that Δ possesses a left adjoint.

One of the more useful statements in category theory is the following:

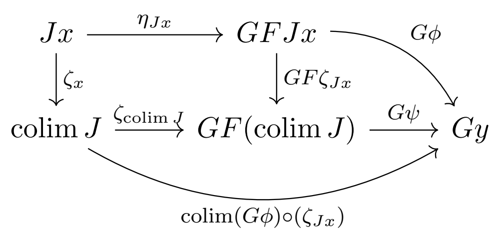
Theorem 5.14 (Adjoints and limits)**.** If $F \dashv G$, $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, and $J : \mathcal{E} \to \mathcal{C}$ is a diagram functor with colimit $\operatorname{colim} J$, then $F(\operatorname{colim} J) = \operatorname{colim}(F \circ J)$. Similarly, if $K : \mathcal{E} \to \mathcal{D}$ is a diagram functor with limit $\lim K$, then $G(\lim K) = \lim(G \circ K)$.

This is more commonly stated as

1. Left adjoints preserve colimits.

2. Right adjoints preserve limits.

Proof. All that is required to show is that $F(\operatorname{colim} J)$ satisfies the representability criterion of $\operatorname{colim}(F \circ J)$. Let ζ be the structure natural transformation $J \to \Delta \operatorname{colim} J$. Then $F(\zeta)$ provides a natural transformation $F \circ J \to \Delta(F \operatorname{colim} J)$. We take these to be the structure maps, and seek to show that, for every $y \in \mathcal{D}$ and natural transformation $\phi : FJ \to \Delta y$, there exists a unique $\operatorname{colim} \phi : F(\operatorname{colim} J) \to y$ such that $(\Delta \operatorname{colim} \phi) \circ \zeta = \phi$. As way of motivation, consider the following diagram, induced after applying G to the proposed diagram in $\mathcal{D}$:



where ψ is the map we want. This gives the recipe: define $\psi : F(\operatorname{colim} J) \to y$ by

$$\psi = \epsilon_y \circ F(\operatorname{colim}(G\phi) \circ (\zeta_{Jx})).$$

Applying F and precomposing with ϵ_y shows that ψ is a map making the diagram commute. Assiming that $\psi' : F(\operatorname{colim} J) \to y$ also made the diagram commute, applying G would land in the diagram above, and there is a unique map from $GF(\operatorname{colim} J) \to Gy$ which commutes with the unique map induced by the map from the colimit of J to Gy by adjointness. But this means that $\psi = \psi'$.

Similarly, right adjoints preserve limits, using an exact parallel of the argument above. ☐

Example 5.15. The group algebra functor $\mathsf{Grp} \to k - \mathsf{Alg}$ is left adjoint to group of units functor $k - \mathsf{Alg} \to \mathsf{Grp}$ which takes any algebra to its group of units.

Example 5.16. The universal enveloping algebra of a Lie algebra L, denoted $U(L)$, provides an example of a left adjoint $U : \mathsf{Lie} \to \mathsf{Alg}$. The right adjoint is the functor that takes any algebra to itself, considered as a Lie algebra with bracket $[x, y] = xy - yx$.

Example 5.17. Let k be a field, and let $k - \mathsf{vect}$ denote the category of its finite-dimensional vector spaces. Then for any $Y \in k - \mathsf{vect}$, the functors $- \otimes Y$ and $Y^* \otimes -$ are both left and right adjoint to each other. This can be seen by computing the unit and counit: the unit takes any object X to

$Y^* \otimes X \otimes Y$ by fixing a basis $\{y_i\}$ and sending each x to $\sum_i y_i^* \otimes x \otimes y$; the counit takes $Y \otimes X \otimes Y^* \to X$ by sending simple tensors $y \otimes x \otimes y^* \to y^*(y) \cdot x$.

Example 5.18 (The GNS construction)**.** [4] Given a C^*-algebra A, the most fundamental tool in its representation theory is the GNS construction, which provides it with a faithful representation on Hilbert space that is universal among all representations. In fact, the GNS is a left adjoint natural transformation in the following sense: let $\mathcal{C}$ be the opposite category of C^*-algebras, sometimes referred to as the category of noncommutative locally compact Hausdorff spaces, and let $\mathcal{D}$ be Cat. Then there are two functors $F, G : \mathcal{C} \to \mathcal{D}$: F is the "states" functor, which takes any C^*-algebra to its category of norm one positive linear functionals, as a poset, and pulls back positive linear functionals along $*$-homomorphisms. G is the "pointed representation" functor, which takes any C^*-algebra to its category of representations, equipped with a norm one point, and $*$-homomorphisms pull back representations. There is a restriction natural transformation $G \to F$, which, for each C^*-algebra A, takes the pointed representation $(\mathcal{H}, \rho : A \to B(\mathcal{H}), \eta)$ to the states $a \mapsto \langle \rho(a), a \rangle$. The GNS construction provides a left adjoint to this natural transformation.

Remark 5.19. For any adjoint functors $F \dashv G$, we are given lots of representatable functors: the functor $\mathcal{D}(F-, y)$ is isomorphic to the functor $\mathcal{C}(-, Gy)$. In fact, if $\mathcal{D}(F-, y)$ is representable for every y, then F is a left adjoint.

Example 5.20. The completion of a metric space is another adjoint, although this time with respect to uniformly continuous functions. Similarly, the Stone-Čech-compactification is an adjoint that goes from locally compact spaces to compact ones.

Example 5.21 (Push-pull formula)**.** Consider the following situation: Assume that we have four categories A, B, C, D, and functors F_1, F_2, G_1, G_2 such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{F_1} & B \\ \downarrow{\scriptstyle G_1} & & \downarrow{\scriptstyle G_2} \\ C & \xrightarrow{F_2} & D \end{array}$$

where all the F_i, G_i are right adjoints to their left adjoints, F_i^*, G_i^*. Then there is a natural transformation $F_2^* G_2 \to G_1 F_1^*$ defined as follows:

$$\begin{aligned} F_2^* G_2 &\cong F_2^* G_2 1_B \\ &\xrightarrow{\eta F_1} F_2^* G_2 F_1 F_1^* \\ &\cong F_2^* F_2 G_1 F_1^* \\ &\xrightarrow{\epsilon_{F_2}} 1_C G_1 F_1^* \\ &\cong G_1 F_1^* \end{aligned}$$

so the natural transformation is exactly

$$(\epsilon_{F_2} 1_{G_1 F_1^*}) \circ (1_{F_2^* G_2} \eta_{F_1}).$$

This may seem like a complicated setup, but it is very common; such a square occurs whenever we have a commuting square of schemes, because the we have both a pushforward and a pullback of quasicoherent sheaves. In fact, this argument works whenever we have a commuting square of dualizable 1-morphisms in a 2-category.

5.3 Reflective and Coreflective Subcategories

One of the ubiquitous concepts in the world of adjoint functors is finding a best approximation to an object inside a smaller subcategory. For example, we can characterize complete metric spaces as being those which process all uniformly continuous functions from a metric space to a complete metric space, or abelianizations as being universal objects that process maps from a non-abelian object into abelian ones.

Definition 5.22. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is called **reflective** if the inclusion functor $\iota : \mathcal{C} \to \mathcal{D}$ has a left adjoint.

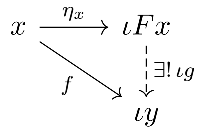
Let us unspool the definitions. Let $\iota : \mathcal{D} \to \mathcal{C}$ be the inclusion, and $F : \mathcal{C} \to \mathcal{D}$ its adjiont. This means that for any objects $x \in \mathcal{C}, y \in \mathcal{D}$, we have

$$\mathcal{C}(x, \iota y) \cong \mathcal{D}(Fx, y).$$

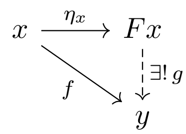
By fullness of $\mathcal{D}$, we can treat F as a functor $\mathcal{C} \to \mathcal{C}$, and restate as

$$\mathcal{C}(x, \iota y) \cong \mathcal{C}(\iota F x, y)$$

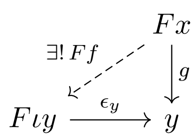
for every $y \in \mathcal{D}$. The isomorphism should be instantiated by the unit: for any morphism $f \in \mathcal{C}(x, \iota y)$, we should have a unique arrow filling in the following diagram:



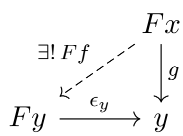
and fullness lets us treat this diagram as



so there is a minimal $\mathcal{D}$-object approximating x. The counit gets phrased as



and using fullness this is



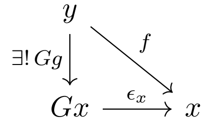
but Fy is actually isomorphic to y. We can prove this via the Yoneda lemma:

$$\begin{aligned} \mathcal{D}(F\iota y, z) &\cong \mathcal{C}(\iota y, \iota z) \\ &\cong \mathcal{D}(y, z) \end{aligned}$$

so the functors $\mathcal{D}(F\iota -, -)$ and $\mathcal{D}(-, -)$ are isomorphic, which shows that $F\iota$ is naturally isomorphic to the identity on $\mathcal{D}$. Because ϵ is the counit on $\mathcal{D}$, ϵ must be an isomorphism (in fact, the isomorphism).

Definition 5.23. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is called **coreflective** if the inclusion functor $\iota : \mathcal{D} \to \mathcal{C}$ has a right adjoint.

Call the adjoint G. The situation here is very similar to the reflective situation, except that Fx is now a maximal $\mathcal{D}$-approximation rather than a minimal one. We have the following diagram:



Example 5.24 (Groupification)**.** Let Mon denote the category of monoids. Then the inclusion functor Grp $\to$ Mon makes Grp a coreflective subcategory: the right adjoint is the functor which takes a monoid to its group of units. It also makes Grp a reflective subcategory: the left adjoint is defined by taking any monoid M to the group with generators M and relations $x_1 \cdot x_2 = (x_1 x_2)$.

Example 5.25 (Stone-Čech Compactification)**.** The Stone-Čech β of a locally compact Hausdorff space is the minimal compact space approximating it. This means that locally compact spaces are a reflective subcategory of locally compact spaces.

Example 5.26. Let $\mathcal{C} = \mathsf{Ab}$, the category of Abelian groups, and $\mathcal{D}$ be the category of 2-torsion Abelian groups: groups for which $2x = 0$ for all x. Then $\mathcal{D}$ is reflective: the left adjoint is the functor $- \otimes \mathbb{Z}_2$.

5.4 Adjoints and Monads

Definition 5.27. A **monad** is a monoid in the category of endofunctors. Explicitly, a monad in a category $\mathcal{C}$ is a functor $T : \mathcal{C} \to \mathcal{C}$ with a natural transformation $\mu : T^2 \to T$ and $\eta : 1_{\mathcal{C}} \to T$ satisfying the following coherence properties:

$$\begin{array}{ccccc} & & T & & \\ & \swarrow^{T\eta} & \downarrow{\scriptstyle 1_T} & \searrow^{\eta T} & \\ T^2 & \xrightarrow[\mu]{} & T & \xleftarrow[\mu]{} & T^2 \end{array}$$

and

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \downarrow{\scriptstyle \mu T} & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

the first diagram expressing that η acts like a two-sided unit for multiplication; the second expressing its associativity.

Theorem 5.28. Every adjoint pair $F \dashv G$, $F \in [\mathcal{C}, \mathcal{D}]$, gives rise to a monad.

Proof. Define $T = GF : \mathcal{C} \to \mathcal{C}$. The unit of the monad is the unit of the adjunction; the multiplication is the map $GFGF \to GF$ by $1_G \epsilon 1_F$, which maps to $G1_{\mathcal{D}}F \cong GF$. We need to show that all the appropriate triangles and squares commute. First,

$$\begin{aligned}
\mu \circ (T\eta) &= 1_G\epsilon 1_F \circ 1_{GF}\eta \\
&= 1_G\epsilon 1_F \circ 1_G 1_F \eta \\
&= 1_G(\epsilon 1_F \circ 1_F \eta) \\
&= 1_G 1_F \\
&= 1_{GF} \\
&= 1_T
\end{aligned}$$

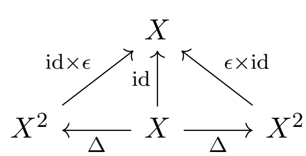
with the other zigzag diagram being used to prove that $\mu \circ (\eta T) = 1_T$. For associativity,

$$\begin{aligned}
\mu \circ (\mu T) &= (1_G\epsilon 1_F) \circ (1_G\epsilon 1_F 1_{GF}) \\
&= 1_G(\epsilon \circ (\epsilon 1_F 1_G))1_F \\
&= 1_G([1_F 1_G \circ \epsilon][\epsilon \circ 1_F 1_G])1_F \\
&= 1_G([\epsilon \circ 1_F 1_G][1_F 1_G \circ \epsilon])1_F \\
&= 1_G(\epsilon \circ 1_F 1_G \epsilon)1_F \\
&= (1_G\epsilon 1_F) \circ (1_{GF} 1_G \epsilon 1_F) \\
\mu \circ (T\mu) &= (1_G\epsilon 1_F) \circ (1_{GF} 1_G \epsilon 1_F)
\end{aligned}$$

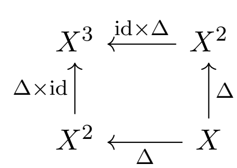
using the commutativity of horizontal and vertical composition of natural transformations. □

Similarly, the other-sided composition of adjoint functors, $\epsilon : FG \to 1_{\mathcal{D}}$, gives rise to a **comonad**.

Definition 5.29. A **comonoid** is the opposite of a monoid. It is an object X together with maps $\epsilon : X \to \{*\}, \Delta : X \to X \times X$ with commutative diagrams



and



Definition 5.30. A **comonad** is a comonoid in the category of endofunctors.

In fact, every monad and comonad come from an adjoint pair, using the Kliesli category.

5.5 Exercises

1. Construct a right adjoint to the forgetful functor $\mathsf{Top} \to \mathsf{Set}$. Define the unit and counit.

2. Show that the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ does not have a right adjoint. [Hint: any such group would have a lifting property that only the trivial group satisfies.]

3. Let $\mathcal{C}$ be the category of k-vector spaces, and define the **Fredholm category** $\mathcal{D} = \mathcal{C}/\mathcal{I}$ as follows:

   i) The objects of $\mathcal{D}$ are the objects of $\mathcal{C}$;

   ii) $\mathcal{D}(V, W) = \mathcal{C}(V, W)/\mathcal{I}(V, W)$, where

   $$\mathcal{I}(V, W) = \{f \in \mathcal{C}(V, W) \mid f(V) \text{ is finite dimensional}\}.$$

   Let F denote the quotient functor $\mathcal{C} \to \mathcal{D}$. Show that F admits neither a left nor a right adjoint. [Hint: limits and colimits over trivial diagrams in $\mathcal{C}$ are the direct sum and product, respectively.]

4. Define the universal property of the metric completion of a metric space in terms of adjoint functors. Again, determine the unit and counit.

5. Let $x \in \mathcal{C}$ be an object, and let $\mathcal{C}/x$ be the category where

   i) Objects are pairs (f, y) with $f : x \to y$;

   ii) Morphisms $(f, y) \to (f', y')$ are maps $g : y \to y'$ such that $gf = f'$.

   $\mathcal{C}/x$ is equipped with a forgetful functor F to $\mathcal{C}$ by forgetting the morphism: $(f, y) \mapsto y$. Show that F admits a right adjoint G if and only if x is an initial object in $\mathcal{C}$, in which case $\mathcal{C}/x \cong \mathcal{C}$. Show that if $\mathcal{C}$ has coproducts, then F has a left adjoint.

6. Let $\mathcal{C}$ be a category, $x \in \mathcal{C}$, and $\mathcal{D} = [\mathcal{C}, \mathsf{Set}]$, the category of covariant functors from $\mathcal{C}$ to Set. Define a functor $\mathrm{ev}_x : \mathcal{D} \to \mathsf{Set}$ by the following rule: $\mathrm{ev}_x(F) = F(x)$, and, if $\eta : F \to F'$, $\mathrm{ev}_x(\eta) = \eta_x$. Show that ev_x is a right adjoint. [Hint: $\mathsf{Nat}(\mathcal{C}(x, -), F) \cong F(x)$.]

7. Prove that Example 5.26 does what it's supposed to.

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