A Convexity Primer

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Credit: Much of this comes from Kadison-Ringrose's text *Fundamentals of the Theory of Operator Algebras* and Bollobás' *Linear Analysis*.

This primer was intended for use alongside Chuck Akemann's 201 course homeworks, but should serve in any discussion of basic convexity theory as applied to function spaces.

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1 Convex Sets

The convexity theory requires real linearity and not complex linearity, even though complex spaces are preferable for spectral theory. Therefore, although we will consider real linear combinations throughout the following section, it is important to bear in mind the results are equally applicable in real and complex vector spaces. Let V be such a vector space.

Definition 1.1. A set $K \subseteq V$ is called *convex* if for every $x, y \in K$ and $t \in (0, 1)$, we have $tx + (1-t)y \in K$.

That is, a convex set is closed under convex combinations. You may have a lot of intuition about convexity from the finite-dimensional case; this is valuable, but relies heavily on the local compactness of these spaces. As a result, this structure has not been helpful in developing intuition for the setting that we most care about—infinite-dimensional function spaces.

Lemma 1.1. *Let $K \subseteq V$ be convex. Then for each $x_1, \ldots, x_n \in K$ and $t_1, \ldots, t_n \in [0, 1]$ with $\sum t_i = 1$, we have*

$$\sum t_i x_i \in K.$$

That is, K is closed under arbitrary finite convex combinations.

Proof. This follows by induction. It's trivially true for $n = 1$ (unlike in most algebraic contexts, we do not let the empty sum denote 0 here; we don't know that 0 is in our set, so we leave it undefined). Assume it is true for $n = k$; then, given $x_1, \ldots, x_{k+1} \in K$ and $t_1, \ldots, t_{k+1} \in [0, 1]$, we show that $\sum_{i=1}^{k+1} t_i x_i \in K$.

If $t_{k+1} = 0$ or 1, we're done; if not, note that

$$\sum_{i=1}^{k} t_i = 1 - t_{k+1}$$

and therefore that

$$\frac{1}{1 - t_{k+1}} \sum_{i=1}^{k} t_i = \sum_{i=1}^{k} \frac{t_i}{1 - t_{k+1}} = 1.$$

In particular, $\sum_{i=1}^{k} \frac{t_i}{1-t_{k+1}} x_i = x \in K$ by the inductive hypothesis.

Now we see that

$$\begin{aligned} \sum_{i=1}^{k+1} t_i x_i =& (1 - t_{k+1}) \sum_{i=1}^{k} \frac{t_i}{1 - t_{k+1}} x_i + t_{k+1} x_{k+1} \\ =& (1 - t_{k+1}) x + t_{k+1} x_{k+1}, \end{aligned}$$

which is therefore in K by definition. So by induction, K contains all of its finite convex combinations. □

An infinite convex combination can be made sense of, but requires some analytic structure we currently lack.

Lemma 1.2. *If $\{K_\alpha\}$ is a family of convex sets, then $K = \bigcap_\alpha K_\alpha$ is convex.*

Proof. This is immediate: if for all α we have $x, y \in K_\alpha$, then by convexity $tx + (1-t)y \in K_\alpha$ for $t \in (0, 1)$. □

Lemma 1.3. *If $\{K_n\}$ is a family of convex sets with $K_0 \subseteq K_1 \subseteq \cdots$, then $K = \bigcup_n K_n$ is convex.*

Proof. This is also immediate: suppose $x, y \in K$. Then for some N we have $x, y \in K_N$, and may apply convexity there to conclude $tx + (1-t)y \in K_N \subseteq K$. □

In fact, we didn't need countability above, just a totally ordered indexing set with $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$.

Definition 1.2. The *convex hull* of a set E, denoted $\operatorname{co} E$, is the smallest convex set containing E. As with most notions of closures or hulls, it is obtained as the intersection of all convex supersets of E. Equivalently, $\operatorname{co} E$ is the set of all convex combinations of elements of E.

Definition 1.3. A *face* of a convex set K is a convex subset $F \subseteq K$ with the property that if for $x, y \in K$ and $t \in (0, 1)$, $tx + (1-t)y \in F$, then $x, y \in F$. An *extreme point* of K is a face that has only one point.

Faces are inaccessible convex subsets: they cannot be entered via nontrivial convex combinations of elements of $K \setminus F$. It is reasonable to imagine proper faces ($F \subset K$) as lying on the boundary of K.

Lemma 1.4. *If G is a face of F, and F is a face of K, then G is a face of K.*

Proof. Let $x, y \in K$ and $t \in (0, 1)$ be such that $tx + (1-t)y \in G$. Then $tx + (1-t)y \in F$, because $G \subseteq F$; F being a face of K then implies that $x, y \in F$. But G is a face of F, so also $x, y \in G$. Hence, G is a face of K. □

In particular, this gives us a natural partial order (by inclusion) on the faces of K.

Definition 1.4. The *dimension* of a convex set K is the dimension of the subspace of V generated by $\{x - y_0 \mid x \in K\}$ for any y_0 in K.

Definition 1.5. A point $x \in K$ is said to be *internal* if for every $y \in V$ there is some $c > 0$ such that for each $0 \leq t < c$, $x + ty \in K$.

Note the order of the quantifiers here: in the familiar case V is equipped with a norm, this is weaker than the statement that x is an interior point. In infinite dimensions, this may be strict.

Example 1.1. Let $V = \ell^1(\mathbb{N})$ and $K = \left\{(a_n) \in \ell^1(\mathbb{N}) \mid \forall k \in \mathbb{N}, |a_k| \le \frac{1}{k}\right\}$. This is convex, as

$$|ta_k + (1-t)b_k| \le t\,|a_k| + (1-t)\,|b_k| \le \frac{1}{k}$$

if $|a_k|\,, |b_k| \le \frac{1}{k}$ and $t \in [0,1]$.

We have (0) internal to K: if $(b_n) \in \ell^1(\mathbb{N})$, then only finitely many terms may exceed $\frac{1}{k}$, else the series would not converge. Thus, $\sup_k k\,|b_k| < \infty$, and dividing (b_n) by this amount lands in K. However, $B_\epsilon((0))$ contains the sequences $\frac{1}{2}\epsilon\delta_k$ for all $k \in \mathbb{N}$ (in particular, for k such that $\frac{1}{2}\epsilon > \frac{1}{k}$), so clearly no ϵ-neighborhood of 0 is contained in K.

We conclude with more examples of computing the faces of convex sets representative of those we're interested in.

Example 1.2. Denote by I the unit interval $[0,1] \subseteq \mathbb{R}$. Then the faces of I are $\{0\}$, $\{1\}$, and I itself.

First, we show that $\{0\}$ is a face. Assume that for some $x, y \in I$ and $t \in (0,1)$ we have $tx + (1-t)y = 0$. Without loss of generality, suppose $x > 0$ (we are uninterested in the case $x = y = 0$). But then $y \ge 0$ implies $tx + (1-t)y \ge tx > 0$: a contradiction.

Similarly, $\{1\}$ is a face: without loss of generality, suppose $x < 1$ in the setting above. Then $tx < t$ implies $tx + (1-t)y < t + (1-t)y \le t + (1-t) = 1$.

Finally, any convex set is trivially a face of itself.

Conversely, assume that $F \subseteq I$ is a face of I. If $F = \{0\}$ or $F = \{1\}$, we're done. If F contains both 0 and 1, it contains their convex hull; as $t1 + (1-t)0 = t$ for $t \in [0,1]$, said hull is $\operatorname{co}\{0,1\} = I$. Otherwise, let $c \in F \setminus \{0,1\} \subseteq (0,1)$. We can write $c = (1-c)0 + c1$ to conclude that 0 and 1 are both in F by the definition of a face. By the argument above, then, $F = I$. Thus, the faces are exactly as claimed.

Example 1.3. Denote by I the unit interval $[0,1]$ in $\mathbb{R}$, and consider $I^n \subseteq \mathbb{R}^n$. Then, considering $\mathbb{R}^n$ as the space of functions $\{f : \{1,\dots,n\} \to \mathbb{R}\}$, the faces of I^n are exactly of the following form:

$$K_{A_0,A_1} = \{f \in \mathbb{R}^n \mid \forall a_0 \in A_0,\ f(a_0) = 0;\ \forall a_1 \in A_1,\ f(a_1) = 1;\ A_0 \sqcup A_1 \subseteq \{1,\dots,n\}\}\,.$$

This requires some work. First, that all such sets are faces: it is straightforward to show that they are convex. For the face property, assume that $g_1, g_2 \in \mathbb{R}^n$ and $t \in (0,1)$ are such that $tg_1 + (1-t)g_2 \in K_{A_0,A_1}$. Then for each $a_0 \in A_0$, $tg_1(a_0) + (1-t)g_2(a_0) = 0$, so $g_1(a_0) + g_2(a_0) = 0$ by nonnegativity. Similarly, for each $a_1 \in A_1$, $g_1(a_1) = g_2(a_1) = 1$. So $g_1, g_2 \in K_{A_0,A_1}$, so it is a face.

For the other direction, let K be a face of I^n. Define

$$A_0 = \{a_0 \in \{1,\dots,n\} \mid \forall f \in K, f(a_0) = 0\}\,,$$

and A_1 similarly. Then $K \subseteq K_{A_0,A_1}$. Let $f \in K_{A_0,A_1}$. We show that there is an element of K which agrees with f at every point—that is, that $f \in K$.

Let $x \in \{1,\dots,n\} \setminus (A_0 \sqcup A_1)$. Then there exists $g \in K$ such that $g(x) \in (0,1)$: $x \notin A_0$, so there is some function with $g(x) \neq 0$, and $x \notin A_1$, so there is some function with $g(x) \neq 1$;

if these functions have values 1 and 0 respectively, any nontrivial convex combination has $g(x) \in (0,1)$. Given this, define the following functions:

$$g_0(i) = \begin{cases} g(i) & i \neq x \\ 0 & i = x \end{cases} \qquad g_1(i) = \begin{cases} g(i) & i \neq x \\ 1 & i = x \end{cases}.$$

Then for any $c \in (0,1)$, $(1-c)g_0(x) + cg_1(x) = c$, and for every $i \in \{1,\ldots,n\} \setminus \{x\}$, $g_0(i) = g_1(i) = g(i)$. Taking $c = g(x)$, we see that g itself is a convex combination of g_0 and g_1, so by the face condition $g_0, g_1 \in K$. Now taking $c = f(x)$, we have the function $(1 - f(x))g_0 + f(x)g_1$ which agrees with f at x and is in K.

To conclude, we show that given $g \in K$, we can choose $\tilde{g} \in K$ with $\tilde{g}(x) \in (0,1)$ and $\tilde{g}(y) = g(y)$ for all $y \neq x$.

This allows us to build a function which agrees with f at each point of $\{1,\ldots,n\}$ (all functions in K agree with f on $A_0 \sqcup A_1$ by definition of those sets) by setting its value one point at a time. That is, $f \in K$, so in fact $K = K_{A_0,A_1}$.

Of particular interest, we have shown that a face of this set is the convex hull of its extreme points. This is a special case of the Krein-Milman Theorem (Theorem 3.1), one of the most important parts of the theory of locally convex topological vector spaces. Unfortunately, not every convex set has this property, as the following fairly general counterexample shows.

Example 1.4. Let $V = \{f : \mathbb{R} \to \mathbb{R}\}$ with the vector space structure induced by pointwise addition and scalar multiplication, and let $K = \{f \in V \mid f(\mathbb{R}) \subseteq I\}$. Then the following face of K is not the convex hull of its extreme points:

$$F = \{f \in K \mid \exists \epsilon > 0 : f((-\epsilon, \epsilon)) = 0\}.$$

These are those functions which vanish on some neighborhood of 0. They again form a face: say $g_1, g_2 \in K$ and $t \in (0,1)$ with $tg_1 + (1-t)g_2 = f \in F$. Fix some $\epsilon > 0$ such that $f((-\epsilon,\epsilon)) = 0$. Then for each $x \in (-\epsilon,\epsilon)$, $f(x) = 0$ implies $g_1(x) = g_2(x) = 0$ just as in the arguments above. So g_1 and g_2 also vanish on $(-\epsilon,\epsilon)$, so F is a face. However, the only set on which every element of F is either 0 or 1 is $\{0\}$. As F is a strict superset of the set of functions which are 0 at 0, it is not the convex hull of its extreme points.

We conclude with a slight generalization of convexity which incorporates the base field of V, which we shall denote as $\mathbb{K}$.

Definition 1.6. A set $B \subseteq V$ is called *balanced* if for every $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, we have $\alpha B \subset B$. That is, B is invariant under scaling by elements of the closed unit disc of $\mathbb{K}$.

Definition 1.7. A set $K \subseteq V$ is called *absolutely convex* if it is both convex and balanced. Equivalently, for every x, y in K and $\lambda, \mu \in \mathbb{K}$ with $|\lambda| + |\mu| \leq 1$, we have $\lambda x + \mu y \in K$.

Similarly to the case with convexity, we may define balanced and absolutely convex hulls by taking intersections over supersets.

While convexity is often the more useful property, certain circumstances require the stronger notion of absolute convexity to be well defined; in particular, it is necessary in the definition of locally convex topological vector spaces.

2 Locally Convex Topological Vector Spaces

In linear analysis, we are often concerned with a Banach space X, with its norm topology. Unfortunately, the norm topology has some serious issues, characterized by the intuition that it has "too few compact sets". To alleviate this, we often introduce weaker topologies. Though they have weaker notions of convergence, these topologies often allow us to make stronger statements. In the following, we will take $\mathbb{K}$ the base field of our Banach space, insisting $\mathbb{R}$ is a subfield of $\mathbb{K}$.

Definition 2.1. The *weak topology* on X has as basic open sets those of the form

$$W_\epsilon(x; \phi_1, \ldots, \phi_n) = \{y \in X \mid |\phi_i(x - y)| < \epsilon\}$$

for $x \in X$, $\phi_1, \ldots, \phi_n \in X^*$, and $\epsilon > 0$. In fact, we only need those sets with $\epsilon = 1$, as $W_\epsilon(x; \phi_1, \ldots, \phi_n) = W_1(x; \epsilon^{-1}\phi_1, \ldots, \epsilon^{-1}\phi_n)$, and a subbasis consists of those sets of the form $W_\epsilon(x; \phi)$ or even $W_1(x; \phi)$ for a single functional. The best geometric picture available for these sets is "slabs" which extend infinitely in all directions except those constrained by the given functionals; this is unwieldy to picture in infinite dimensions, but indicates the important fact that these sets are *not* norm-bounded.

A sequence x_n converges to x in the weak topology (also *converges weakly*, denoted $x_n \xrightarrow{w} x$) if for all $\phi \in X^*$ we have the numerical sequence $\phi(x_n - x) \to 0$. As discussed in Appendix A, we can equivalently specify the topology using the net convergence $x_\gamma \xrightarrow{w} x$ defined by $\phi(x_\gamma - x) \to 0$ for all $\phi \in X^*$.

Equivalently, the weak topology is the initial topology on X with respect to the functionals comprising X^*: it is the weakest topology such that all such functionals are continuous. Here, it is important to remember that X^* was originally defined as the space of norm-continuous (equivalently, bounded) functionals on X (equipped with its norm topology). Thus, in making this definition, we begin with a norm topology on X, select the norm-continuous functionals, then declare a new topology on X with respect to these functionals. This does not in general correspond to the initial topology corresponding to the algebraic dual X' (while this can be defined, the weak topology described here is generally more useful, as it does encode some information about the original norm structure).

Equivalently, the weak topology is the seminorm topology with respect to the family of seminorms defined by $||x||_\phi = |\phi(x)|$ for $\phi \in X^*$. In fact, we can restrict to the seminorms induced by ϕ in the unit sphere of X^* (with respect to the dual norm topology on X^*). As we shall see, this phrasing is of especial theoretical importance.

Finally, the weak topology can be described diagramatically: a map f from a topological space A to X is weakly continuous if and only if for every $\phi \in X^*$, $\phi \circ f$ is continuous. That is, the following diagrams commute for all $\phi \in X^*$:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ & {\scriptstyle \phi\circ f}\searrow & \downarrow{\scriptstyle \phi} \\ & & \mathbb{K} \end{array}$$

The weak topology is generally weaker (or coarser) than the norm topology. The Hahn-Banach Theorem (our Theorem 2.2) will show that it continues to be Hausdorff.

Definition 2.2. The *weak star* (alternately *weak** or w^*) topology on X^* has as basic open sets those of the form

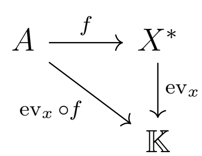
$$W_\epsilon^*(\phi; x_1, \ldots, x_n) = \{\psi \in X^* \mid |\phi(x_i) - \psi(x_i)| < \epsilon\}$$

for $\phi \in X^*$, $x_1, \ldots, x_n \in X$, and $\epsilon > 0$. Again, we only need to consider $\epsilon = 1$ by rescaling (and linearity), and a subbasis is given by those sets with only a single evaluation point as data.

A sequence ϕ_n converges to ϕ in the weak star topology (also *converges weakly star* and denoted $\phi_n \xrightarrow{w^*} \phi$) if for all $x \in X$ we have the numerical sequence $\phi_n(x) \to \phi(x)$. An equivalent specification for the topology is the net convergence defined analogously.

Equivalently, the weak star topology is the initial topology with respect to the functions on X^* given by evaluation at a fixed $x \in X$ (that is, the functions $\mathrm{ev}_x : X^* \to \mathbb{K}$ defined by $\mathrm{ev}_x(\phi) = \phi(x)$), and the seminorm topology with respect to the seminorms defined by $||\phi||_x = |\mathrm{ev}_x(\phi)| = |\phi(x)|$.

Finally, a map $f : A \to X$ from a topological space A is weak star continuous if and only if for every $x \in X$ the following diagram commutes:



Since the mapping $x \mapsto \mathrm{ev}_x$ gives a canonical injection of X into X^{**}, this is an initial topology with respect to only a subset of the (continuous) functionals on X^*. It should be unsurprising that the weak star topology is therefore generally weaker than the weak topology on X^*, although it is again also Hausdorff. Spaces for which the weak and weak star topologies are the same are called *reflexive*, and are of particular interest.

A crucial property of the weak star topology is that it preserves the compactness of the unit ball. This allows us to continue to appeal to (norm) boundedness to extract weak star convergent subsequences, despite the fact that the weak star topology is not locally compact (the unit ball is not a neighborhood of the origin, as all basic open sets are norm-unbounded).

Theorem 2.1 (Banach-Alaoglu)**.** *The closed unit ball of X^* (with respect to the dual norm) is compact in the weak star topology.*

Proof. For each $x \in X$, define $D_x = \{z \in \mathbb{K} \mid |z| \leq ||x||\}$, observing that this is compact in $\mathbb{K}$. Now define

$$D = \prod_{x \in X} D_x$$

(with the product topology). By Tychonov's theorem, this is compact.

Let $B_1^* \subset X^*$ be the unit ball of X^* (under the dual norm), and define $T : B_1^* \to D$ by $T(\phi)_x = \phi(x)$. This is injective, as if $T(\phi) = T(\psi)$, then $\phi(x) = \psi(x)$ for all $x \in X$, so ϕ and ψ are the same functional. It is weak star continuous: a subbasis of D consists of those sets $U_{x,z,\epsilon} = \{Z \in D \mid |Z_x - z| < \epsilon\}$, so $T^{-1}(U_{x,z,\epsilon}) = \{\phi \in B_1 \mid |\phi(x) - z| < \epsilon\}$. If ϕ is a member of this set, then $W_{\epsilon'}(\phi; x) \subset T^{-1}(U_{x,z,\epsilon})$ for $\epsilon' < \frac{1}{2}(\epsilon - |\phi(x) - z|)$ by the Triangle Inequality. Thus, $T^{-1}(U_{x,z,\epsilon})$ is open and T is continuous.

Now consider the partial inverse $S : \operatorname{Im} T \to B_1^*$ ($S = T|_{\operatorname{Im} T}^{-1}$). For any $z_0 \in \mathbb{K}$ and $\epsilon > 0$, the open set $V = \{z \in \mathbb{K} \mid |z - z_0| < \epsilon\}$, we have (considering ev_x restricted to B_1^*)

$$\begin{aligned}(\mathrm{ev}_x \circ S)^{-1}(V) =& S^{-1} \circ \mathrm{ev}_x^{-1}(V)\\ =& T\left(\{\phi \in B_1^* \mid |\phi(x) - z_0| < \epsilon\}\right)\\ =& \{Z \in D \mid |Z_x - z_0| < \epsilon\},\end{aligned}$$

which is open in the product topology. Thus, the diagram

$$\begin{array}{ccc} \operatorname{Im} T & \xrightarrow{S} & B_1^* \\ & {\scriptstyle \mathrm{ev}_x \circ S}\searrow & \downarrow {\scriptstyle \mathrm{ev}_x} \\ & & \mathbb{K} \end{array}$$

commutes, so S is weakly star continuous as well. Thus, T is a homeomorphism onto its image, so all that remains to show B_1^* is compact is to show $T(B_1^*) \subset D$ is closed.

Let $Z \in \overline{T(B_1^*)} \subseteq D$. We can think of this as a map $\phi : X \to \mathbb{K}$ by $\phi(x) = Z_x$; we seek to show it is linear, continuous, and of norm bounded by 1. The last follows from the definition of D: as $|Z_x| < ||x||$, we must have $|\phi(x)| \leq ||x||$ for all x, so $||\phi|| \leq 1$ (this also gives continuity). Armed with this, for any $x, y \in X$ and $\epsilon > 0$ we can choose $\psi \in B_1^*$ such that simultaneously $|\phi(x) - \psi(x)| < \frac{1}{3}\epsilon$, $|\phi(y) - \psi(y)| < \frac{1}{3}\epsilon$, and $|\phi(x+y) - \psi(x+y)| < \frac{1}{3}\epsilon$. Then

$$\begin{aligned}|\phi(x+y) - \phi(x) - \phi(y)| =& |\phi(x+y) - \psi(x+y) + \psi(x) - \phi(x) + \psi(y) - \phi(y)|\\ \leq& |\phi(x+y) - \psi(x+y)| + |\phi(x) - \psi(x)| + |\phi(y) - \psi(y)|\\ <& \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon,\end{aligned}$$

so ϕ is linear. This shows that in fact $Z \in T(B_1^*)$, in particular $Z = T(\phi)$. Thus, $T(B_1^*)$ is closed, hence compact by compactness of D, hence B_1^* is compact, as T is a homeomorphism with respect to the weak star topology. □

A proof very similar to this one will show the compactness of the unit ball in the space of operators with a similar topology we will define there, the weak operator topology.

The important commonality between the norm, weak, and weak star topologies is that they make X or X^* into an object known as a locally convex topological vector space. In fact, such spaces are slightly more general; this generalization happens to exactly capture the kinds of topologies useful to functional analysis, so we define them here.

Definition 2.3. A *topological vector space* V (over the base field $\mathbb{K}$) is a vector space endowed with a topology τ such that addition and $\mathbb{K}$-multiplication are continuous maps.

This establishes the basic compatibility of the algebraic and topological structures of V. For example, it provides a uniform structure on V: if a function from V is continuous at 0 (or any point), it is continuous on all of V by translation. Our model for a uniform structure is $\mathbb{R}^n$, where the topology is generated by the family of open balls about the origin, which scale and translate to give a local description of general open sets. The correct generalization is to find a similarly structured neighborhood basis of the origin; absolute convexity of these basic open sets provides a convex structure we can use. We additionally require the basic open sets to satisfy the following definition.

Definition 2.4. A neighborhood $A \subset V$ of the origin is *absorbent* if for all $x \in V$ there is some $t \in \mathbb{R}$ positive such that $x \in tA$.

Definition 2.5. A *locally convex* topological vector space is one in which the origin has a neighborhood basis of absolutely convex absorbent sets.

The right way to think about the local convexity condition is that it asserts "open balls are convex". In fact, one can show that it is precisely the condition that allows us to discuss open balls in the first place: a locally convex topological vector space is precisely a topological vector space whose topology is generated by a family of seminorms.

The locally convex setting is the right place to do much of functional analysis. A principal justification for this statement is that in any locally convex Hausdorff topological vector space, we have the following powerful theorem:

Theorem 2.2 (Hahn-Banach)**.** *If Y and Z are disjoint non-empty convex subsets of a locally convex topological space V, and Y is open, then there exists a functional $\rho \in V^*$ such that for each $y \in Y$ and $z \in Z$ we have*

$$\mathfrak{Re}\rho(y) > k \geq \mathfrak{Re}\rho(z)$$

for some real $k \geq 0$. Further, if in addition at least one of Y and Z is compact, there are real numbers a, b and a $\rho \in V^*$ such that for each $y \in Y$, $z \in Z$,

$$\mathfrak{Re}\rho(y) \geq a > b \geq \mathfrak{Re}\rho(z).$$

As a consequence of this, we have the following more standard statement of the theorem in the Banach space setting.

Corollary 2.3. *Given a (not necessarily closed) subspace U of a Banach space X, and a functional ϕ defined on U, there exists a functional $\psi \in X^*$ extending ϕ. Furthermore, this extension may be chosen such that $||\psi|| = ||\phi||$.*

Theorem 2.2 takes a lot of work to prove, but it can be found in Kadison-Ringrose.

Example 2.1. $\mathbb{K}^n$ with its norm topology is locally convex. In fact, a nice theorem states that there is only one locally convex linear topology on $\mathbb{K}^n$.

Example 2.2. A Banach space X equipped with its norm topology, its weak topology, or any weak star topology that it may happen to have, is locally convex.

Example 2.3. Let X and Y be Banach space, and let $\mathcal{B}(X, Y)$ be the space of continuous operators from X to Y. Then the following linear topologies are all locally convex:

- The topology induced by the *operator norm*, given by the norm $\|T\|_{X \to Y} = \sup_{\|x\|_X = 1} \|Tx\|_Y$

- The *strong operator topology*, characterized by the convergence

$$T_\gamma \xrightarrow{sot} T \iff \forall x \in X,\ \|T_\gamma x - Tx\|_Y \to 0.$$

- The *weak operator topology*, characterized by the convergence

$$T_\gamma \xrightarrow{wot} T \iff \forall x \in X, \phi \in X^*,\ \phi(T_\gamma x - Tx) \to 0.$$

Example 2.4. Let (Ω, μ) be a non-atomic finite-measure space, and define a metric on the measurable functions $f : \Omega \to \mathbb{R}$ which have finite values everywhere but a set of measure 0 by

$$d(f, g) = \int_\Omega \frac{|f - g|}{1 + |f - g|}\, d\mu.$$

This is a translation-invariant metric with respect to which addition and scalar multiplication are continuous, and therefore a topological vector space; call this space X. X fails to be locally convex. Assume that $\phi : X \to \mathbb{R}$ is continuous. Then $\phi^{-1}(-1, 1)$ is a convex open set in X containing 0. We claim that $\phi^{-1}(-1, 1)$ is all of X.

Let $g \in X$ be arbitrary. For any partition $\sqcup_{i=1}^n K_i = \Omega$, letting $\chi_i = \chi_{K_i}$, we have

$$g = \sum_{i=1}^n \chi_i g = \frac{1}{n} \sum_{i=1}^n n\chi_i g.$$

By continuity of ϕ, there is some $\delta > 0$ such that if $d(f, 0) < \delta$, then $|\phi(f)| < 1$. By non-atomicity of Ω, there is such a partition with $\mu(K_i) < \delta$ for each i, and a simple bounding argument shows that this means that $|\phi(n\chi_i g)| < 1$:

$$\begin{aligned} d(n\chi_i g, 0) &= \int_\Omega \frac{|n\chi_i g|}{1 + |n\chi_i g|}\, d\mu \\ &= \int_\Omega n\chi_i \frac{|g|}{1 + |n\chi_i g|}\, d\mu \\ &= \int_{K_i} \frac{|ng|}{1 + |ng|}\, d\mu \\ &\leq \mu(K_i) \\ &< 1 \end{aligned}$$

So by convexity of the absolute value function, $|\phi(g)| < 1$. But this was true for arbitrary $g \in X$, so $\phi \equiv 0$. So the continuous dual of X is only the 0 function, which would be impossible in a locally convex space by Hahn-Banach.

This proof is especially cute because we never had to investigate the topology of the space, but nonetheless concluded facts about its topology.

This metric is usually called the metric of *convergence in measure*.

3 Krein-Milman

One of the jewels of the theory of locally convex topological vector spaces is the Krein-Milman theorem.

Theorem 3.1 (Krein-Milman)**.** *Let V be a locally convex topological vector space, and $K \subseteq V$ a compact convex set. Then $K = \overline{\text{co}}E$, where E is the set of extreme points of K.*

First, we show the existence of an extreme point. We apply Zorn's Lemma to the the collection of closed faces of K, ordered by reverse inclusion.

Let $A = \{\text{closed faces of } K\}$, with $F_1 \preceq F_2 \iff F_1 \supseteq F_2$. Let $\{F_i\}_{i \in I}$ be an ascending chain in A. It has an upper bound: $F = \cap_I F_i$. This is nonempty because it is the intersection of a decreasing family of nonempty closed sets in a compact set; it is convex because the intersection of arbitrarily many convex sets, so long as the intersection is nonempty, is convex.

It is a face of K: let $x, y \in K$, $t \in (0,1)$ such that $tx + (1-t)y \in F$. Then for each F_i, $x, y \in F_i$ by the face condition, so $x, y \in \cap_I F_i$, so $x, y \in F$.

It is the intersection of closed sets, and therefore closed; it is a closed subset of a compact set, so compact.

Now that every chain has an upper bound, Zorn's Lemma produces a maximal element, say F_0. Assume that F_0 is not a point. Pick two points $x \neq y \in F_0$. By Hahn-Banach, there is a functional ρ such that $\operatorname{Re}\rho(x) > \operatorname{Re}\rho(y)$. Because ρ is continuous and F_0 is compact, $\operatorname{Re}\rho$ attains a maximum value c. Now examine

$$G = \{z \in F_0 \,|\, \operatorname{Re}\rho(z) = c\}$$

Certainly $y \notin G$. G is the inverse image of a closed set under continuous function, so G is closed. Moreover, G is a face of F_0: if $r, s \in F_0$, $t \in (0,1)$ such that $tr + (1-t)s \in G$, then

$$\begin{aligned} \operatorname{Re}\rho(tr + (1-t)s) &= c \\ t \cdot \operatorname{Re}\rho(r) + (1-t) \cdot \operatorname{Re}\rho(s) &= c \end{aligned}$$

Assume, without loss of generality, that $r \notin G$. Then

$$\begin{aligned} t \cdot \operatorname{Re}\rho(r) + (1-t) \cdot \operatorname{Re}\rho(s) &< t \cdot c + (1-t) \cdot \operatorname{Re}\rho(s) \\ &\leq t \cdot c + (1-t)c \\ &= c \end{aligned}$$

a contradiction. So $r, s \in G$, so G is a face of F_0. By Lemma 1.2, G is a face of K, and because $y \notin G$, $G \subsetneq F_0$. So G is a strictly smaller nonempty closed face (closed because it is compact in a Hausdorff space), contradicting the maximality of F_0. So F_0 must consist of a point.

This produces an extreme point for us. Now let $E = \{\text{extreme points of } K\}$. Consider $\overline{co}E$. It is closed, compact, convex. Assume that there is some $y \in K \setminus \overline{co}E$. Again, by Hahn-Banach, we can separate these convex sets by a functional ρ. Pick ρ so that $\operatorname{Re}\rho(y) > \operatorname{Re}\rho(\overline{co}E)$. Again, the compactness of K implies that $\operatorname{Re}\rho$ attains a maximum on K; the inverse image of this maximum is a face by the above arguments. By the above argument on the existence of an extreme point, there must be some extreme point in $K \setminus \overline{\text{co}\, E}$. But E contained all the extreme points, a contradiction. So $\overline{\text{co}}E = K$.

4 Convex Spaces and Convex Functions

5 Krein-Milman in Dual Spaces

Question. Is my space a dual space?

The fact that the weak star topology is compact on the unit ball means that if my space could be given a weak star topology, the unit ball would be the closed convex hull of its extreme points. In particular, it would need to have extreme points. For many locally convex topological vector spaces, this is not the case, ruling out the possibility of them being dual spaces.

Example 4.1. $C_0(\mathbb{R})$, $C[0,1]$, with the Banach structure coming from the supremum norm Consider the vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, vanishing at infinity. It is a Banach space when equipped with the supremum norm. The unit ball has no extreme points, however:

Let $\|f\| \leq 1$. There is some compact set K such that $\forall x \in \mathbb{R} \setminus K$, $|f(x)| < \frac{1}{2}$. Pick a smooth bump function ψ with the support of ψ contained in $\mathbb{R} \setminus K$ and $\|\psi\|_\infty = \frac{1}{4}$; then $|f(x) + \psi(x)| \leq 1$, $|f(x) - \psi(x)| \leq 1$, so $f + \psi$ and $f - \psi$ are both in the unit ball. But $\frac{1}{2}(f + \psi) + \frac{1}{2}(f - \psi) = f$, so f is not an extreme point. So $C_0(\mathbb{R})$ is not a dual space.

The argument for $C[0,1]$ is different. The unit ball of $C[0,1]$ does contain extreme points, but the entire thing is not a closed convex hull of its extreme points. I assert that the functions $f \equiv 1$ and $f \equiv -1$ are extreme points: doing just $f \equiv 1$, if $f = tg_1 + (1-t)g_2$, then for each x, $tg_1(x) + (1-t)g_2(x) = 1$. But 1 is an extreme point of $[-1,1]$, so $g_1(x) = g_2(x) = 1$ for each x. Similarly, $f \equiv -1$ is an extreme point.

Now assume that $f \in C[0,1]$ is neither constantly 1 nor constantly -1. Assume that for each $c \in (-1,1)$, $f(x) \neq c$; then $f(x) \in \{-1,1\}$ for each $x \in [0,1]$. But $[0,1]$ is connected and f is continuous, so $f(x)$ must map strictly to $\{1\}$ or to $\{-1\}$, contradicting it not being constantly either. So there is some $c \in (-1,1)$ such that for some $x_0 \in [0,1]$, $f(x_0) = c$.

Now let $r = \min\{|1-c|, |c-(-1)|\}$. This is positive, because c is neither 1 nor -1. By continuity of f, there is some $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - c| < \frac{r}{2}$. In particular, $\max\{|f(x) - 1|, |f(x) + 1|\} \leq \frac{r}{2}$ by the triangle inequality. So taking a smooth bump function ψ supported inside $(x_0 - \delta, x_0 + \delta)$ with $\|\psi\|_\infty = \frac{r}{4}$, we again have $|(f + \psi)(x)| \leq 1$, $|(f - \psi)(x)| \leq 1$, and $f = \frac{1}{2}(f + \psi) + \frac{1}{2}(f - \psi)$. So the only extreme points are the functions 1 and -1.

If $C[0,1]$ were a dual space, then the closure of their convex combinations in the supremum norm would be all of the unit ball. But $t1 + (1-t)(-1) = t - 1 + t = 2t - 1$, so their convex combinations are constant functions, which are one-dimensional. Because there is only one locally convex linear topology on a one-dimensional space, these convex combinations are homeomorphic to $[0,1]$, which is compact and cannot be dense in any Hausdorff space. So $C[0,1]$ is not a dual space.

Example 4.2. Let $T : X \to Y$ be a continuous bijective linear mapping of Banach spaces, and $T^* : Y^* \to X^*$ map the extreme points of the unit ball in Y^* to the extreme points of the unit ball in X^* bijectively. Then T is an isometric isomorphism.

I show that $\|Tx\|_Y \leq \|x\|_X$; the other direction follows from the fact that T^{-1} has all the properties that T does. We need the following two statements:

$$\|x\| = \sup_{\phi \in (X^*)_1} |\phi(x)|$$
$$\|Tx\| = \sup_{\psi \in (Y^*)_1} |\psi(Tx)|$$

I claim that T^* maps the unit ball of Y^* to the unit ball of X^*. Let $\psi \in (Y^*)_1$. There is a convergent net of convex combinations of extreme points, say $f_\gamma \xrightarrow{w^*} \psi$. T^* is weak star continuous (prove this) so $T^*(f_\gamma) \xrightarrow{w^*} T^*(\psi)$. But $T^*(f_\gamma) = f_\gamma \circ T$ is a convex combination of extreme points, and therefore in the unit ball. By the weak star closure of the unit ball, $T^*(\psi)$ is in the unit ball as well. This completes the proof.

Example 4.3. Extreme points in von Neumann algebras
Let H be a Hilbert space, $\mathcal{B}(H)$ the space of bounded operators on H with its standard involution $*$. We call a $*$-closed subalgebra A of $\mathcal{B}(H)$ a *von Neumann algebra*, or *W^*-algebra*, if A is closed in the weak operator topology. By an argument very similar to that of the Banach-Alaoglu theorem, the unit ball of $\mathcal{B}(H)$ is compact in the weak operator topology (the unit ball in H is weakly compact). Therefore, the unit ball of A, equipped with the weak operator topology, is also compact. By Krein-Milman, it should have extreme points. Unfortunately, these extreme points aren't things we generally want to deal with. We restrict to the following two cases: the positive cone of A, and the self-adjoint cone of A.
Claim: extreme points of the unit ball of the entire thing are the isometries, extreme points of the self-adjoint cone are projections.

Example 4.4. Recovering the Riemann integral
For X a locally compact Hausdorff topological space, the dual space of $C(X)$ is $M(X)$, the regular countably additive Borel measures on X. (This is the Riesz-Markov theorem.) With respect to the weak star topology, the unit ball and the positive unit ball of $M(X)$ are compact. Taking $X = [0, 1]$, and noting that the Lebesgue measure corresponds exactly to the Riemann integral, we can reconstruct the Riemann integral as exactly the weak star limit of

$$m_n = \sum_{i=1}^{n} \frac{1}{n} \delta_{\frac{i}{n}}$$

which is the standard method to evaluate the Riemann integral for a continuous function.

6 The Pettis Integral

A Nets

A net is used for fishing.

B Initial Topologies

Oftentimes, given a space X and maps $f_i : X \to Y_i$, where the Y_i are topological spaces, we wish to define the weakest topology on X such that each of these maps are continuous.

C Seminorm Topologies

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