Homework 5 Solutions

3.1 (a) Show that the probability that an individual of disease age $t$ is still infected is $e^{-\gamma t}$.

Individuals have a probability of $\delta t$ of leaving the infected class over the time interval $[t, t+\delta t]$. If we denote by $I(t)$ the fraction of individuals in the disease class at time $t$ then at time $t+\delta t$ we have

$$ I(t+\delta t) = I(t) - \delta t \cdot I(t) $$

As $\delta t \to 0$, this can be reexpressed as an ODE

$$ \frac{dI}{dt} = -\gamma I $$

which has solution

$$ I(t) = I(0) e^{-\gamma t} $$

If we start with all individuals in the disease class at time 0, then $I(0) = e^{-\gamma t}$ is the fraction of individuals of age $t$ that remain in the disease class. The fraction of individuals in the disease class at time $t$ is the probability that an individual remains in the disease class by the frequency interpretation of probabilities.

(b) Show that the mean time spent in the infective class is $\frac{1}{\gamma}$.

The probability that an individual has disease age $\tau$ between $t$ and $t+\delta t$ is the probability it leaves the disease class at some time between $t$ and $t+\delta t$, which is given by

$$ I(t) - I(t+\delta t) $$
that is the probability that the individual is still in the disease class at time $t$ but is not in the disease class at time $t + \delta t$.

For $\delta t$ small we can approximate this probability by

$$I(t) - I(t + \delta t) = \frac{I(t) - I(t + \delta t)}{\delta t} \delta t = \frac{dI(t)}{dt} \delta t$$

$$\frac{dI(t)}{dt} = -\gamma e^{-\gamma t} \delta t$$

thus the probability that an individual exits the disease class at time $t$ is

$$\gamma e^{-\gamma t} \delta t$$

To find the mean time (average time) an individual exits the disease class we sum over all the exit times weighted by the probability of the exit time,

$$\text{average } T = \sum_k T_k \gamma e^{-\gamma t} \delta t$$

where $T_k$ is a sample from the interval $[k\delta t, (k+1)\delta t]$, $T_k = k\delta t$, $k \in \mathbb{N}$

Again for $\delta t$ small we can make an approximation that becomes exact as $\delta t \to 0$. The sum is the Riemann sum of the integral

$$\text{average } T = \int_0^\infty T \gamma e^{-\gamma t} dt = \left[-\frac{e^{-\gamma t}}{\gamma}\right]_0^\infty - \int_0^\infty e^{-\gamma t} dt$$

$$= \left[-\frac{e^{-\gamma t}}{\gamma}\right]_0^\infty + \frac{1}{\gamma} \int_0^\infty e^{-\gamma t} dt$$

$$= \left[-e^{-\gamma t}\right]_0^\infty + \frac{1}{\gamma} \left[-e^{-\gamma t}\right]_0^\infty$$

$$= 0 + \frac{1}{\gamma} \left[0 - (-1)ight]$$

$$= \frac{1}{\gamma}$$
3.2 Let the probability that an individual of disease age \( \sigma \) is still in the infective class be \( f(\sigma) \).

(a) The incidence \( i \) of a disease is the rate at which new cases occur (show that

\[
i(t) = \beta N \int_{0}^{\infty} f(\sigma) i(t - \sigma) d\sigma
\]

when \( t \geq N \).

An intuitive argument for this expression can be made by considering the definitions of \( i(t) \) and \( f(\sigma) \).

\[
i(t) = \frac{\# \text{ of new cases per unit of time}}{\delta t} = \frac{\# \text{ of new cases occurring at times } [t, t + \delta t]}{\delta t}
\]

\[
I(t) = \frac{\# \text{ of infectives at time } t}{\delta \sigma} = \frac{\# \text{ new cases occurring at times } [\sigma_k, \sigma_k + \delta \sigma]}{\delta \sigma} \times \text{(probability individual is still infected at time } t)
\]

\[
= \int_{\sigma_k}^{\sigma_{k+1}} i(t - \sigma) f(\sigma) d\sigma\delta \sigma
\]

\( f(\sigma) \delta \sigma = \text{probability individual infected } \sigma \text{ units of time ago is still infected} \)

For \( \delta t \) small and \( \delta \sigma \) small we have

\[
I(t) \approx \left( \int_{0}^{\infty} f(\sigma) i(t - \sigma) d\sigma \right) \delta t
\]

thus

\[
i(t) \delta t = \beta N \int_{0}^{\infty} f(\sigma) i(t - \sigma) d\sigma \delta t
\]

since both sides are of first order in \( \delta t \) we have

\[
i(t) = \beta N \int_{0}^{\infty} f(\sigma) i(t - \sigma) d\sigma \quad \text{as } \delta \sigma \to 0.
\]
(b) \( i(t) = \beta N \int_0^\infty \sigma e^{-\sigma} i(t-\sigma) \, d\sigma \)

\( f(\sigma) = \sigma e^{-\sigma} \) is the probability that an individual of disease age \( \sigma \) is still infected.

\[
i(t) = i_0 e^{rt} \quad \text{then}
\]

\[
i_0 e^{rt} = i_0 \beta N \int_0^\infty \sigma e^{-\sigma} e^{-(r+\sigma)t} \, d\sigma
\]

\[
= i_0 \beta N \left[ -\frac{1}{r+\sigma} e^{-\sigma} \right]_0^\infty e^{rt}
\]

\[
= i_0 \beta N \left[ \frac{1}{r+\sigma} \right] e^{rt}
\]

\[
\Rightarrow \frac{1}{\beta N} = \frac{1}{r+\sigma} \quad , \quad \frac{r+\sigma}{\beta N} = \gamma
\]

\[
v = \beta N \gamma - \gamma = (\beta N - 1) \gamma.
\]

(c) \( r = R_0 \gamma - \gamma \)
3.6 Let $R_0w \ll 1$

(a) Show $\frac{dw}{dt} = 1 - w - e^{-R_0w}$ may be approximated by

$$\frac{dw}{dt} = (R_0 - 1)w(1 - \frac{w}{w_1})$$

where $w_1 = \frac{2(R_0 - 1)}{R_0^2}$

The Taylor expansion of $e^x = 1 + x + \frac{x^2}{2} + \cdots$ yields

$$e^{-R_0w} = 1 - R_0w + \frac{R_0^2w^2}{2} + \cdots$$

$R_0w \ll 1$ so we may truncate after a finite number of terms to obtain an approximate ($k = 2$),

$$\frac{dw}{dt} \approx 1 - w - 1 + R_0w - \frac{R_0^2w^2}{2} = (R_0 - 1)w - \frac{R_0^2w^2}{2} = (R_0 - 1)w(1 - \frac{wR_0}{2(R_0 - 1)}) = (R_0 - 1)w(1 - \frac{w}{w_1})$$

(b) Deduce Kermack and McKendrick's second threshold theorem that is $\frac{(R_0 - 1)}{R_0} \ll 1$

then the size of the epidemic is approximately

$$w_1 = \frac{2(R_0 - 1)}{R_0^2}$$

Since the approximate equation is logistic, the population monotonically approaches the non-trivial steady-state. For an epidemic to occur requires $R_0 > 1$, so $w(t) \leq w_1 = \frac{2}{R_0}(\frac{R_0 - 1}{R_0}) \ll 1$
This suggests that the approximation made in the equation for \( w \) is valid throughout the course of the disease since \( R_0 w \leq 2 \left( \frac{R_0 - 1}{R_0} \right) \ll 1 \) holds.

In the long time limit, the population tends to a disease-free steady state and the infected class \( w(t) \rightarrow w_1 \), the carrying capacity of the logistic equation. Since there is no death in the model, the size of the epidemic is equal to the number who have recovered when the population reaches the disease-free steady state. Therefore, the size of the epidemic is approximately

\[
\frac{w_1}{R_0} = 2 \left( \frac{R_0 - 1}{R_0} \right).
\]
(c) Show that the incidence of death (removal from population) for small epidemics follows a sech² curve, \( w(t) \), satisfies the logistic equation

\[
\frac{dw}{dt} = r w \left( 1 - \frac{w}{K} \right)
\]

where \( r = (R_0 - 1) \), \( K = w_t \).

This equation has the exact solution

\[
w(t) = \frac{Kw_0 e^{rt}}{K - w_0 + w_0 e^{rt}}
\]

The rate of change of \( w \) is given by

\[
\frac{dw}{dt} = \frac{rKw_0 e^{rt}}{K - w_0 + w_0 e^{rt}} - \frac{rKw_0^2 e^{rt}}{(K - w_0 + w_0 e^{rt})^2}
\]

\[
= \frac{rKw_0^2 e^{rt}(K - w_0 + w_0 e^{rt})}{(K - w_0 + w_0 e^{rt})^2} - \frac{rKw_0^2 e^{rt}}{(K - w_0 + w_0 e^{rt})^2}
\]

\[
= \frac{(rKw_0)(K - w_0)e^{rt}}{(K - w_0)(e^{-rt} + w_0e^{rt} + \alpha (K - w_0)w_0)}
\]

\[
= \left[ \frac{(K - w_0)(w_0 K)}{4(K - w_0)w_0} \right] \left[ \frac{4}{w_0(K - w_0)} e^{-rt} + \frac{w_0}{(K - w_0)e^{rt} + \alpha} \right]
\]

\[
= \left[ \frac{(K - w_0)(w_0 K)}{4(K - w_0)w_0} \right] \left[ e^{-(rt + \ln \left( \frac{w_0}{K - w_0} \right))} + e^{rt + \ln \left( \frac{w_0}{K - w_0} \right)} \right]
\]

\[
= \left[ \frac{(K - w_0)(w_0 K)}{4(K - w_0)w_0} \right] \text{sech}^2 (rt + \ln \left( \frac{w_0}{K - w_0} \right))
\]
\[ \frac{du}{dt} = \frac{b}{y+b} (1-u) - R_0 uv \]
\[ \frac{dv}{dt} = (R_0u - 1) v \]
\[ \frac{dw}{dt} = \frac{v}{y+b} - \frac{b}{y+b} w \]

\[ u^* = \frac{1}{R_0}, \quad v^* = \frac{b}{y+b} \left( 1 - \frac{1}{R_0} \right) \]
\[ w^* = 1 - u^* - v^* \]

are steady-states verified by substitution into the right hand side of the equations above.

The linearization of the equations above is:
\[ f(u,v,w) = \frac{b}{y+b} (1-u) - R_0 uv \]
\[ g(u,v,w) = (R_0u - 1) v \]
\[ h(u,v,w) = \frac{v}{y+b} - \frac{b}{y+b} w \]

\[ \frac{\partial f}{\partial u} = \frac{-b}{y+b} - R_0v, \quad \frac{\partial f}{\partial v} = -R_0u, \quad \frac{\partial f}{\partial w} = 0 \]
\[ \frac{\partial g}{\partial u} = R_0v, \quad \frac{\partial g}{\partial v} = (R_0u - 1), \quad \frac{\partial g}{\partial w} = 0 \]
\[ \frac{\partial h}{\partial u} = 0, \quad \frac{\partial h}{\partial v} = \frac{v}{y+b}, \quad \frac{\partial h}{\partial w} = -\frac{b}{y+b} \]

If \( R_0 > 1 \), the steady-state \((u^*, v^*, w^*)\) is biologically relevant and we shall investigate its stability.

If \( R_0 < 1 \) then \( v^* < 0 \) and the steady-state is not relevant.
When $R_0 > 1$ the Jacobian is

$$J_{w^*, r^*, w^*} = \begin{bmatrix}
-\frac{b}{\gamma + b} & -R_0 v^* & -1 & 0 \\
R_0 v^* & 0 & 0 & 0 \\
0 & \frac{r}{\gamma + b} & -\frac{b}{\gamma + b} & 0 \\
\end{bmatrix}$$

$$\det J = -\frac{b}{\gamma + b} [R_0 v^*]$$

$$\text{tr } J = -\frac{b}{\gamma + b} - R_0 v^*$$

In 3-D Routh–Hurwitz criteria are not be used in the same way as in the 2-D problem and we shall instead look at the eigenvalues of $J$.

$$\rho(u) = \begin{vmatrix}
\lambda - \alpha & 1 & 0 \\
-R_0 v^* & \lambda & 0 \\
0 & \frac{r}{\gamma + b} & \lambda + \frac{b}{\gamma + b} \\
\end{vmatrix}$$

$$= (\lambda + \frac{b}{\gamma + b}) \left( (\lambda - \alpha) \lambda + R_0 v^* \right)$$

$$\lambda^2 - \alpha \lambda + R_0 v^* = 0$$

$$\lambda = \alpha \pm \sqrt{\alpha^2 - 4R_0 v^*}$$

$$\alpha < 0$$

therefore $\Re(\lambda_k) < 0$ for all eigenvalues and the steady-state is stable.