Final Project Descriptions

A brief summary of each of the final projects can be found below. Please consult with your classmates to form groups of no more than 4 for each project. When you have decided on a group, please e-mail me your names. The projects will be assigned in the order in which I receive requests. A more detailed description of the questions that are to be addressed, along with a few suggestions will be provided to each group.

Project I: Portfolio Analysis

The primary approach discussed in the course to quantitate asset characteristics, and investor preferences of risk and return when constructing portfolios was Markowitz’s theory. To apply the approach in practice requires that one specify for each of the asset returns the mean, variance, and covariance. One basic approach to determining these quantities is to analyze historical data for the assets. There are of course a number of drawbacks to using only historical data, but this is a good starting point in modeling many types of assets. In this project you will be given a collection of historic time series for the prices of a number of assets. Using the methods discussed in class you will write software to numerically estimate the mean, variance, and covariance of the assets. For given investor preferences your software should also compute the optimal portfolio as specified by the Markowitz optimization problem. In the project you will then explore how well the theory characterizes these investments and the role a risk-free asset plays in determining the optimal portfolio.

- Download the historical time series data for the assets and related matlab codes to read the data. The time series are of 10 assets and span a time of 10 years.

- The empirical expected returns over one month can be computed from the following formulas.

Let the return over the period \(\tau\) be denoted by:

\[
r_i(t, \tau) = \frac{s_i(t + \tau) - s_i(t)}{s_i(t)}.
\]

The mean and variance of the \(i^{th}\) asset can be estimated by:

\[
\mu_i = \frac{1}{N} \sum_{k=1}^{N} r_i(t_k, \tau)
\]

and

\[
\sigma_i^2 = \frac{1}{N-1} \sum_{k=1}^{N} (r_i(t_k, \tau) - \mu_i)^2
\]
where \( t_k = k\tau \) and \( N = (T - \tau)/\tau \).

The covariance can be estimate by

\[
\sigma_{i,j} = \frac{1}{N} \sum_{k=1}^{N} (r_i(t_k, \tau) - \mu_i)(r_j(t_k, \tau) - \mu_j).
\]

In each formula above you should use \( \tau = 1/1000 \) which corresponds to approximately 1/3 of a day.

- Compute the mean returns and variances of each of the assets.
- Compute the covariance matrix \( V \) from the time series data of the first 4 assets using the estimates above.
- Assume that only the first four assets are correlated and construct the full \( 10 \times 10 \) covariance matrix \( V \) for all 10 assets. Use the previously computed covariance matrix and the variances of the remaining assets, do not compute the full matrix as this is too expensive. In other words, this will have the \( 4 \times 4 \) covariance matrix of the first 4 assets in the upper left corner and the remaining non-zero entries will simply be the diagonal entries corresponding to the variances of the remaining assets as estimated above.
- Use the numerical method developed in class to give the weights for the portfolio with smallest variance having expected return \( \mu = 0.2 \).
- Give the optimal portfolio consisting of only the first 2 assets, 5 assets, and 10 assets.
- Construct a plot of the estimated variance of the return for the portfolio verses the number of assets included in the portfolio. In other words, use your code to construct portfolios incorporating the first \( k \) assets and plot the variance. Are there any trends? Which portfolio has the least variance? Why?
- For the random asset prices included with the downloaded package, compute a histogram of the returns over the period \( \tau \) for each of the assets. Do you think the mean and variance are sufficient to characterize the assets in each case? Why?
- For the random asset prices, compute a histogram of the return for the optimal portfolio having \( \mu = 0.2 \), and estimate the mean and variance.
- How does the average return and empirical variance compare with the theory for this portfolio?
Project II: Binomial Tree Market Model

The binomial tree approach to modeling markets was introduced in the class. While the use of binomial trees is not especially efficient in numerically pricing options, the model has the advantage of being conceptually fairly straightforward. In class it was shown how portfolios can be constructed which replicate the value of an option. We also showed that the principle of no-arbitrage ensured that an equivalent probability can be associated with the price movements in the market which give the option value in terms of a discounted risk-neutral expected payoff. In this project you will write software to construct the hedging portfolios for a given option and empirically explore some properties of the risk-neutral probability.

• Write a code which computes the value of an option with a given payoff function \( f(s) \) at maturity time \( T \) for a given spot price \( s_0 \). Assume that the binomial market has \( N \) periods with prices given by

\[
s_0 u^k d^{N-k}
\]

at each node, where \( k \) is the number of up steps required to reach the node.

• For

\[
\begin{align*}
d & = e^{(r - \frac{\sigma^2}{2}) \delta t - \sigma \sqrt{\delta t}} \\
u & = e^{(r - \frac{\sigma^2}{2}) \delta t + \sigma \sqrt{\delta t}}
\end{align*}
\]

compute the value of the risk-neutral probability

\[
q = \frac{e^{r \delta t} - d}{u - d}
\]

where \( r = 0.02 \) and \( \sigma = 0.1 \). As \( \delta t \to 0 \), plot \( q \) verses \( \delta t \). To what value does \( q \) appear to converge?

• Compute the value of a call option and a put option with spot price \( s_0 = 30 \), strike price \( K = 30 \), volatility \( \sigma = 0.20 \), and maturity \( T = 1 \) year when \( r = 0.02 \). Give the computed option value when using the following number of periods \( N = 1, 3, 10, \) and 40.

• Use your code to compute the option prices verses the number of periods \( N \) and plot the results. Does the option value appear to converge? For what value of \( N \)?

• Write a code which performs a random walk on the tree by taking an up step with the risk-neutral probability \( q \) and a down step with probability \( 1 - q \).
• Compute the option value using the Monte-Carlo method and compare this with your other results when the number of periods \( N = 20 \) and the number of samples \( M = 10,000 \). Do you get a similar value for the option? Be sure to discount the expected payoff.

• Compute the value of the put and the call for a random interest rate model where each period the interest rate \( r \) is a uniform random variable falling in the interval \([0.01, 0.03]\). Use the Monte-Carlo method to compute the option value. Does this change the option value relative to the deterministic interest rate model?

**Project III: Black-Scholes PDE**

From the theory of stochastic processes it was found for a stock having lognormal price dynamics the value of the option obeys a partial differential equation. Partial differential equations can be solved numerically by spatial and temporal discretization onto a grid. In this project you will use a numerical scheme for the Black-Scholes PDE to price European options. You will then use the exact solution formulas for calls and puts to explore the accuracy and convergence of the numerically estimated values of the option.

• Using the following numerical scheme for the Black-Scholes formula:

\[
\frac{V_j^{n+1} - V_j^n}{\Delta t} + \frac{1}{2} \sigma^2 s_j^2 \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta s^2} - r s_j V_j^{n+1} - V_j^n - \frac{1}{2} \sigma^2 s_j^2 - rV_j^n = 0.
\]

estimate the value function \( V(s, t) \) of a call option at time \( t = 0 \) with maturity \( T = 0.025 \), when the strike price is \( K = 2 \), \( r = 0.02 \), \( s_0 = 2 \), and \( \sigma = 0.2 \). Use a grid with \( \Delta s = 0.1 \), \( \Delta t = 0.0025 \).

• The option value is expressed by the Black-Scholes Formula. Use the matlab code to compute this value.

• With \( \Delta t = 0.25 \Delta x^2 \), plot the error of the numerical solution verses the nearly exact solution obtained from the Black-Scholes Formula as \( \Delta x \to 0 \).

• Write a code to solve the standard heat equation using the scheme:

\[
\frac{u_j^{n+1} - u_j^n}{\Delta \tau} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}.
\]

• Use the change of variable transformation given in class which relates the Black-Scholes PDE to the standard heat equation to obtain the value of the option above.

• Compare the solutions you obtained by these means with the previous two methods.
Project IV: Monte-Carlo Approach to Derivative Pricing

It was found in class that for both the binomial model and the lognormal model of stock price dynamics the value of an option could be expressed as a discounted risk-neutral expected payoff. One approach to valuing the option is to evaluate the expected payoff directly by generating many instances of the random stock price values, under the appropriate probability distribution. In this project you will use a random walk approximation to the lognormal stock price dynamics and value options which both depend on the stock price at maturity and on the value of the stock price at multiple times before maturity. The Monte-Carlo approach will be used whereby you will generate many instances of the paths taken by the random stock price to estimate the expected payoff of the option. You will also be asked to explore the accuracy and convergence of the estimates.

- Use the following scheme to generate approximate trajectories of Brownian motion $w(k\delta t) \approx \tilde{w}_k$:

  \[
  \begin{align*}
  \tilde{w}_0 &= 0 \\
  \tilde{w}_{k+1} &= \tilde{w}_k + \sqrt{\delta t}Z_k
  \end{align*}
  \]

  where $Z_k$ are independent standard Gaussian random variables generated for each index $k$.

- To generate trajectories which approximate the lognormal stock dynamics $s(k\delta t) \approx \tilde{s}_k$ use the scheme:

  \[
  \begin{align*}
  \tilde{s}_0 &= s_0 \\
  \tilde{s}_{k+1} - \tilde{s}_k &= r\tilde{s}_k\delta t + \sigma \tilde{s}_k (\tilde{w}_{k+1} - \tilde{w}_k).
  \end{align*}
  \]

- Compute a histogram of the stock prices at maturity when $s_0 = 2$, $\delta t = 0.1$, $\sigma = 0.2$, $r = 0.02$ and $T = 1$.

- Compute a histogram of the logarithm of the stock prices at maturity $T = 1$.

- Compute the value of a European call and European put option using the Monte-Carlo method, in which you compute multiple trajectories of the lognormal stock dynamics and average the payoff of the option to estimate the expected payoff. Show your estimate of the option value versus the number of samples $N$ you used and give a plot of your results. For what value of $N$ does your estimate appear to have converged?

- Compare the value with the results of the Black-Scholes formula. Give the log-log plot of the error. At what rate does the error appear to converge?

- Compute the value of an Asian style call and put option which has payoff which averages the stock price over the interval $[0, T]$.
• Compute the value of an Asian style call and put option which has a time dependent volatility $\sigma(t) = (t/T)^{0.1} + (1 - t/T)^{0.3}$. How does this change the value of the option?