Homework III (Solutions)

1) Call option has payoff \( f(S_T) = (S_T - K)_+ \)

\[
\begin{align*}
&100 \\ &80 \\
\end{align*}
\]

\( K = 90 \)
\( T = 1 \)
\( r = 0.0453102 \)

Replicating Portfolio:

\( f_1 = (80 - 90)_+ = 0 \), find \( w_1, w_2, w_\ast, w_\ast_2 \)

\( f_2 = (120 - 90)_+ = 30 \),

\( w_1 S_T + w_2^\ast = f(S_T) \)

\( w_1 80 + w_2^\ast = 0 \)

\( w_1 120 + w_2^\ast = 30 \)

\( \Rightarrow w_1^\ast = \frac{3}{4}, w_2^\ast = -60 \)

Value at time 0:

\[
V(f) = w_1^\ast 100 + w_2^\ast e^{-rT}
\]

\[
= \frac{3}{4} 100 + (-60) e^{-0.0453102}
\]

\[
= 20.4545 \text{ value of call}
\]

b) Put option has payoff \( f(S_T) = (K - S_T)_+ \)

\[
\begin{align*}
&K = 90, \quad f_1 = (90 - 80)_+ = 10 \\
&f_2 = 0 \\
&w_1 80 + w_2^\ast = 10 \\
&w_1 120 + w_2^\ast = 0
\end{align*}
\]

\( \Rightarrow w_1^\ast = \frac{1}{4}, \quad w_2^\ast = 30 \)

Value at time 0:

\[
V(f) = \frac{1}{4} 100 + 30 e^{-rT} = 2.2727 \text{ value of put}
\]
c) \( f(S_T) = (S_T - K)^+ \), \( K = 90 \)

\[
\begin{align*}
f_1 &= (80 - 90)^+ = 0 \\
f_2 &= (120 - 90)^+ = 30
\end{align*}
\]

\[
\begin{align*}
w_1 S_0 + w_2 &= 100 \quad \Rightarrow \quad w_1 = 20, \quad w_2 = -1500 \\
w_1 120 + w_2 &= 900 \quad \Rightarrow \quad w_2 = 30
\end{align*}
\]

Value at time 0:

\[
V(t) = 20 \cdot 100 + (-1500 \cdot e^{-rT}) = 635.36 \quad \text{value of option}
\]

\[
\text{d) Trinomial Model:}
\]

\[
\begin{align*}
100 & \quad r = 0.039421 \\
104 & \quad T = 1 \\
80 &
\end{align*}
\]

Call option strike \( K = 104 \), \( f_1 = 0 \), \( f_2 = 0 \), \( f_3 = 16 \)

Replicating Portfolio:

\[
\begin{align*}
w_1 S_0 + w_2 &= 0 \\
w_1 104 + w_2 &= 0 \\
w_1 120 + w_2 &= 16
\end{align*}
\]

No solution for \( w_1, w_2 \).

The payoff cannot be replicated by a portfolio in this market.
We now determine the optimal portfolios in the sense
\[
\max w_1 s_0 + w_2 e^{-rT} \leq V(f) \leq \min w_1 s_0 + w_2 e^{-rT} \\
w_1 s_T + w_2 \leq f(s_T) \quad w_1 s_T + w_2 \geq f(s_T)
\]

We first look for \( \min, \max \) (critical point of \( P = w_1 s_0 + w_2 e^{-rT} \) inside the constraint domain

\[
\frac{\partial P(w_1, w_2)}{\partial w_1} = s_0 \neq 0, \quad \frac{\partial P(w_1, w_2)}{\partial w_2} = e^{-rT} \neq 0
\]

thus no interior \((w_1, w_2)\) are optimal for either constraints,

\[
\begin{align*}
&\; w_1 s_T + w_2 \leq f(s_T) \quad \text{nor} \quad (w_1, w_2) \quad w_1 s_T + w_2 \geq f(s_T)
\end{align*}
\]

Note, that the constraints specify a convex set of the form \( \mathbb{W} \):

\[
\begin{align*}
(w_1 s_0 + w_2 &\leq 0 \\
w_1 10 + w_2 &\leq 0 \\
w_1 120 + w_2 &\leq 16
\end{align*}
\]

(\#) shows no extrema exist in the interior.

A similar calc. to (\#) can be used to show no extrema occur on the open line segments between the vertices. Therefore, \( \max \) is achieved at vertex
For the max $w_1 \leq 104 + w_2 \leq 16$ we have

$v_1 : \begin{cases} w_{104} + w_2 = 0 \\ w_{104} + w_2 = 0 \end{cases} \Rightarrow w_1 = 0, w_2 = 0$

$\Rightarrow v_1 = (0, 0)$

$w_1 + 104 + w_2 \leq 16$ must hold to make $v_1$ feasible

Clearly, this holds for $w_1 = 0$, so $v_1$ is feasible.

$v_2 : \begin{cases} w_{104} + w_2 = 0 \\ w_{104} + w_2 = 16 \end{cases} \Rightarrow w_1 = \frac{4}{10}, w_2 = -\frac{32}{10}

\Rightarrow v_2 = \left(\frac{4}{10}, -\frac{32}{10}\right)$

$w_1 + 104 + w_2 \leq 0$ must hold to make $v_2$ feasible

$\frac{4}{10} \cdot 104 - 32 > 0$, so $v_2$ is not feasible.

$v_3 : \begin{cases} w_{104} + w_2 = 0 \\ w_{104} + w_2 = 16 \end{cases} \Rightarrow w_1 = 1, w_2 = -104

\Rightarrow v_3 = (1, -104)$

$w_1 + 80 + w_2 \leq 0$ required for feasibility.

$1 + 80 + (-104) < 0$ shows that $v_3$ is feasible.

The max $v_3$ with $w_1 + \ldots$ occurs at $t$
\[
\max \quad w_1 s_0 + w_2 e^{-rT} = 2.869 \times 10^{-5}
\]

\[
w_1 s_T + w_2 \leq f(s_T)
\]

Now for the min, all the inequalities are reversed
\[
\begin{align*}
&\text{if } w_1 s_0 + w_2 \geq 0 \\
&\text{if } w_1 10^4 + w_2 \geq 0 \\
&\text{if } w_1 10 + w_2 \geq 16
\end{align*}
\]

\[v_1 = (0, 0) \text{ is in feasible}
\]

\[v_3 = (\frac{4}{10}, -32) \text{ is now feasible}
\]

\[v_8 = (1, -104) \text{ is now infeasible}
\]

Thus min occurs at \(v_3\) with
\[
\min \quad w_1 s_0 + w_2 e^{-rT} = \frac{4}{10} \cdot 10^4 - 32 \cdot e^{-rT}
\]

\[
w_1 s_T + w_2 \geq f(s_T)
\]

\[= 9.2308
\]

Therefore, no arbitrage imposes the following bound on the price of the call options:
\[
2.869 \times 10^{-5} \leq V(f) \leq 9.2308
\]
Similar solution technique holds for part (e) or (f).

2) a) \[ q_0 \quad 140 \quad f_5 \]

\[
\begin{array}{c}
q_0 \\
140 \\
100 \quad (1-q) \\
80 \quad 100 \\
60 \quad f_3
\end{array}
\]

Call: Strike \( K = 100 \).

\[
q_0 = \frac{e^{rt} \cdot \text{Down} - \text{Down}}{\text{Up} - \text{Down}} = \frac{1000 \cdot 0.5505 - 80}{120 - 80}
\]

\[
= \frac{2202}{40} = 0.5505
\]

\((1-q_0) = 0.4495\)

\(q_1 = 0.5505, \quad (1-q_1) = 0.4494\)

\[f(s_T) = (s_T - K)^+\]

\[f_3 = (60 - 100)^+ = 0\]

\[f_4 = (100 - 100)^+ = 0\]

\[f_5 = (140 - 100)^+ = 40\]

\[f_0 < f_1 < f_4 = 0\]

\[f_2 = -a\]
\[ f_2 = e^{-r\delta t} \left[ q_1 f_5 + (1-q_1) f_4 \right] \]
\[ = e^{-r\delta t} \left[ q_1 f_4 + (1-q_1) 0 \right] \]
\[ = 21,980 \]
\[ f_1 = e^{-r\delta t} \left[ q_4 f_4 + (1-q_4) f_5 \right] \]
\[ = 0 \]
\[ f_0 = e^{-r\delta t} \left[ q_0 f_4 + (1-q_0) f_1 \right] \]
\[ = e^{-r\delta t} \left[ q_0 \times 1,5840 + 0 \right] \]
\[ = 11,8605 \quad \text{value of the call option.} \]

b) Change the client \( f_0 = 11,8605 \) and execute the hedging strategy given by the replicating portfolios.

Choose \( w_1, w_2 \) so that
\[
\begin{align*}
  \{ w_1 \times 120 + w_2 = 21,980 \} & \Rightarrow w_1 = \frac{21,980}{40} \\
  \{ w_1 \times 80 + w_2 = 0 \} & \Rightarrow w_2 = -21,980 \times 0
\end{align*}
\]

This portfolio is constructed at time 0.

If the stock goes down, we have an option worth \( f_1 = 0 \), so the new replicating portfolio has \( w_1 = 0 \), \( w_2 = 0 \).
If the stock goes up we have an option worth $f_2 = 21.9801$. We then construct a replicating portfolio with
\[
\begin{align*}
\langle w_1, 140 + w_2 = 40 \rangle & \Rightarrow w_1 = 1 \\
\langle w_2, 100 + w_2 = 0 \rangle & \Rightarrow w_2 = -100
\end{align*}
\]

(c) No, look carefully at the value of the portfolio from the last period and the newly specified replicating portfolio. They have the same value at each node so the hedge is self-financing.

d)
\[
\begin{align*}
f_2 & < 40 \\
f_2 & < 0
\end{align*}
\]

We construct a replicating portfolio,
\[
\begin{align*}
\langle w_1, 140 + w_2 = 40 \rangle & \Rightarrow w_1 = 1, \ w_2 = -100 \\
\langle w_1, 100 + w_2 = 0 \rangle & \Rightarrow w_1 = 0, \ w_2 = 0
\end{align*}
\]
For the first period we have:

\[ f_2 = 21,5840 \]

\[ f_0 \quad f_1 = 0 \]

\[ w_1 = \frac{21,5840}{40} \]
\[ w_2 = -2 \cdot 21,5840, \quad f_0 = 11,6467 \]

An adjustment need only be made in the case that the stock goes up.

This gives

\[ w(1) - w(0) = 1 - \frac{21,5840}{40} = 0,4604 \quad \text{transaction cost} \]

which is the amount of stock we must purchase.

The only other transaction cost occurs when we set up the portfolio,

\[ w(0) = \frac{21,5840}{40} \quad \text{at time } 0. \]

We now state all these costs in dollars at time 0.

Since one cannot be certain if the stock will go up or down, we shall charge the client the greater of the two transaction costs over the first period. This gives a cost in time \( t = 0 \) dollars of

\[ \text{Cost}_1 = e^{-5t} \cdot 0,4604 = 0,5113 \]

The second transaction cost occurs at time 0:

\[ \text{Cost}_2 = \frac{21,5840}{40} = 0,5394 \]
This requires we charge the client

\[ V_{\text{no trans.}, (f)} + \text{total transaction costs} = 11.6467 + 0.9909 = 12.6376 \]

Provided the hedging strategy above is optimal with respect to handling transaction costs in the market, no arbitrage requires the option be worth 12.6376.
Problem 4:

a) "contract value exp. ret. theory"
   = expected winnings - expected losses
   = $2 \times 10^7 \times 10^{-2} - 0 = 2 \times 10^5 = $200,000.

If offered $200,000 in cash vs. the contract, I'd take the cash which is certain. There is a 99% chance you end up with nothing under the contract, and I value a certain, but still substantial payoff, rather than an uncertain payoff which on average is the same. In this circumstance my preferences indicate I am not a risk-neutral investor. In other words, I would demand some premium in order to take a large risk. The expected returns theory used above treats the value of both payoffs above the same.

b) "contract value" = $2 \times 10^9 \cdot 10^{-3} - 0
   = 2 \times 10^6 = $2,000,000.

c) Personally, I would not enter this contract since $2,000,000 offers one a pretty comfortable life-style. While the contract payoff is large, it occurs with almost nil probability, 0.1%. I would not be
will to sacrifice my millionaire life-style just to be a billionaire knowing I may lose everything with probability 99.99%.

d) Again, the exorbitant amount earned in this context does not seem to mean much to an individual who has a current level of comfortable wealth, when this risks losing it all.

e) "contract value" = 2\times10^4 \times 10^{-4} = 0
= 2 \times 10^7 = 20,000,000.

This contract exceeds $20,000,000 in value under our expected returns (risk-neutral) theory which would have predicted that an individual would purchase such a contract "mis-priced" at $2,000,000.

These examples illustrate other factors need to be taken into account in such circumstances.